AP® Exam Practice Questions for Chapter 9

1. $y(t) = 20t - \frac{5}{2}t^2 \qquad x(t) = 2t^3 - 12t^2$

$y'(t) = 20 - 5t \qquad x'(t) = 6t^2 - 24t$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{20 - 5t}{6t^2 - 24t}$$

$$0 = \frac{20 - 5t}{6t^2 - 24t}$$

$$0 = 20 - 5t$$

$$5t = 20$$

$$t = 4$$

$$x(4) = 2(4)^3 - 12(4)^2 = -64$$

$$y(4) = 20(4) - \frac{5}{2}(4)^2 = 40$$

The particle is at rest when $t = 4$ at the point $(-64, 40)$.

So, the answer is B.

2. $A = 2 \cdot \frac{1}{2}\int_0^{\pi} \left(\sin^2 \theta\right)^2 \, d\theta$

$$= \int_0^{\pi} \sin^4 \theta \, d\theta$$

So, the answer is C.

3. $x = r \cos \theta = \theta \cos \theta \qquad y = r \sin \theta = \theta \sin \theta$

$$\frac{dx}{d\theta} = -\theta \sin \theta + \cos \theta \qquad \frac{dy}{d\theta} = \theta \cos \theta + \sin \theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\theta \cos \theta + \sin \theta}{-\theta \sin \theta + \cos \theta}$$

At $\theta = -\frac{\pi}{2}$,

$$\frac{dy}{dx} = \frac{-\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2}\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}$$

So, the answer is B.

4. $x = 2t^2 + 3t \qquad y = t^3 + 4t^2$

$$\frac{dx}{dt} = 4t + 3 \qquad \frac{dy}{dt} = 3t^2 + 8t$$

At $t = 2$,

$$\frac{dy}{dx} = \frac{dx/dt}{dy/dt} = \frac{3t^2 + 8t}{4t + 3} = \frac{3(2)^2 + 8(2)}{4(2) + 3} = \frac{28}{11}.$$

At $t = 2$, $x(2) = 2(2)^2 + 3(2) = 14$ and $y(2) = (2)^3 + 4(2)^2 = 24.$

An equation of the tangent line is

$$y - 24 = \frac{28}{11}(x - 14).$$

So, the answer is B.

5. $x = r \cos \theta$

$$= (3 + 5 \sin \theta)(\cos \theta)$$

$$= 3 \cos \theta + 5 \sin \theta \cos \theta$$

$$\frac{dx}{d\theta} = -3 \sin \theta + 5 \sin \theta(-\sin \theta) + 5(\cos \theta)(\cos \theta)$$

$$= -3 \sin \theta - 5 \sin^2 \theta + 5 \cos^2 \theta$$

$$y = r \sin \theta = (3 + 5 \sin \theta)(\sin \theta) = 3 \sin \theta + 5 \sin^2 \theta$$

$$\frac{dy}{d\theta} = 3 \cos \theta + 10 \sin \theta \cos \theta$$

At $\theta = 0$,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \cos \theta + 10 \sin \theta \cos \theta}{-3 \sin \theta - 5 \sin^2 \theta + 5 \cos^2 \theta} = \frac{3(1) + 0}{0 - 0 + 5(1)} = \frac{3}{5}.$$

So, the answer is B.

6. $$\begin{aligned}\mathbf{v}(t) &= \langle x'(t), y'(t)\rangle \\ &= \left\langle 3t^2 - \tfrac{1}{2}, 3(3t-2)^2(3)\right\rangle \\ &= \left\langle 3t^2 - \tfrac{1}{2}, 9(3t-2)^2\right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{a}(t) &= \langle 6t, 18(3t-2)(3)\rangle \\ &= \langle 6t, 54(3t-2)\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{a}(1) &= \langle 6(1), 54(3\cdot 1 - 2)\rangle \\ &= \langle 6, 54\rangle\end{aligned}$$

So, the answer is D.

7. At $t = 4$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{y'(t)}{x'(t)} \\ &= \frac{2e^{-4(t)} + 3}{t\cos t} \\ &= \frac{2e^{-4(4)} + 3}{4\cos 4} \\ &\approx -1.147.\end{aligned}$$

So, the answer is B.

8. (a) At $t = 3$, the speed is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(e^{-t^2+1} - 2\right)^2 + \left(3\sqrt{25 - t^2}\right)^2}$$

$$= \sqrt{\left(e^{-(3)^2+1} - 2\right)^2 + \left(3\sqrt{25 - (3)^2}\right)^2} \approx 12.165.$$

2 pts: { 1 pt: expression for speed; 1 pt: answer }

(b) $$\int_0^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^5 \sqrt{\left(e^{-t^2+1} - 2\right)^2 + \left(3\sqrt{25 - t^3}\right)^2}\, dt$$

$$\approx 59.725$$

2 pts: { 1 pt: integral (arc-length); 1 pt: answer }

Reminder: Write down the appropriate definite integral before numerically approximating it on your calculator. No additional work is needed.

(c) $$y(3) = y(0) + \int_0^3 \frac{dy}{dt}\, dt = 3 + \int_0^3 3\sqrt{25 - t^2}\, dt$$

$$= 3 + 3\left[\frac{1}{2}\left(t\sqrt{25 - t^2} + 25 \arcsin \frac{t}{5}\right)\right]_0^3 \approx 45.131$$

3 pts: { 1 pt: integral; 1 pt: uses initial condition; 1 pt: answer }

Reminder: Write down the appropriate definite integral before numerically approximating it on your calculator. No additional work is needed.

(d) $\dfrac{dy}{dx} = \dfrac{3\sqrt{25 - t^2}}{e^{-t^2+1} - 2}$ is undefined when

$$e^{-t^2+1} - 2 = 0$$
$$e^{-t^2+1} = 2$$
$$-t^2 + 1 = \ln 2$$
$$-t^2 = \ln 2 - 1$$
$$t = \pm\sqrt{1 - \ln 2}$$
$$t \approx 0.5539.$$

$$\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \left\langle e^{-t^2+1} - 2,\ 3\sqrt{25 - t^2}\right\rangle$$

$$\mathbf{a}(t) = \left\langle -2te^{-t^2+1},\ 3\left(\frac{1}{2}\right)\left(25 - t^2\right)^{-1/2}(-2t)\right\rangle = \left\langle -2te^{-t^2+1},\ -\frac{3t}{\sqrt{25 - t^2}}\right\rangle$$

$$\mathbf{a}\left(\sqrt{1 - \ln 2}\right) = \left\langle -2\sqrt{1 - \ln 2}\,e^{-\left(\sqrt{1-\ln 2}\right)^2+1},\ \frac{-3\sqrt{1 - \ln 2}}{\sqrt{25 - \left(\sqrt{1 - \ln 2}\right)^2}}\right\rangle$$

$$\approx \langle -2.216,\ -0.334\rangle$$

At $t = \sqrt{1 - \ln 2} \approx 0.5539$, $\mathbf{a}(t) = \langle -2.216, -0.334\rangle$.

2 pts: { 1 pt: finds where $dx/dt = 0$; 1 pt: answer }

Note: Stating $\mathbf{a}(0.5539) \approx \langle x''(0.5539), y''(0.5539)\rangle$ is sufficient justification. The components of this vector can then be approximated on your calculator using the given $x'(t)$ and $y'(t)$ without showing further work.

Reminders: Round each answer to at least three decimal places to receive credit on the exam. Use more than three decimal places in the intermediate step of finding the t-value in part (d).

Use "≈" rather than an equal sign in presenting these approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

9. (a) $\frac{dx}{dt} = \ln\left[5 + (1 + t)^3\right]$ $\qquad \frac{dy}{dt} = 4t - 3t^2$

$$\frac{d^2x}{dt^2} = \frac{1}{5 + (1 + t)^3} \cdot 3(1 + t)^2 \qquad \frac{d^2y}{dt^2} = 4 - 6t$$

$$= \frac{3(1 + t)^2}{5 + (1 + t)^3}$$

$$\mathbf{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \left\langle \frac{3(1 + t)^2}{5 + (1 + t)^3}, 4 - 6t \right\rangle$$

$$\mathbf{a}(2) = \left\langle \frac{3(1 + 2)^2}{5 + (1 + 2)^3}, 4 - 6(2) \right\rangle = \left\langle \frac{27}{32}, -8 \right\rangle$$

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\left(\ln\left[5 + (1 + t)^3\right]\right)^2 + \left(4t - 3t^2\right)^2}$$

At $t = 2$, the speed is

$$\sqrt{\left(\ln\left[5 + (1 + 2)^3\right]\right)^2 + \left[4 \cdot 2 - 3(2)^2\right]^2} \approx 5.2926.$$

2 pts: {1 pt: finds acceleration vector at $t = 2$; 1 pt: finds speed at $t = 2$}

Notes: Stating $\mathbf{a}(2) = \langle x''(2), y''(2) \rangle$ is sufficient justification. The components of this vector can then be approximated on your calculator using the given $x'(t)$ and $y'(t)$ without showing further work.

Writing Speed $= \sqrt{(x'(2))^2 + (y'(2))^2}$ is sufficient justification. This value can then be approximated on your calculator without showing further work.

Reminders: Round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting these approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

(b) $x(0) + \int_0^3 \left(\frac{dx}{dt}\right) dt = 4 + \int_0^3 \ln\left[5 + (1 + t)^3\right]$

$$\approx 4 + 9.0279 = 13.0279$$

2 pts: {1 pt: integral; 1 pt: answer}

Reminders: Write down the appropriate definite integral before numerically approximating it on your calculator. No additional work is needed.

Round each answer to at least three decimal places to receive credit on the exam.

(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t - 3t^2}{\ln\left[5 + (1 + t)^3\right]}$

At $t = 3$, $\frac{dy}{dx} = \frac{4(3) - 3(3)^2}{\ln\left[5 + (1 + 3)^3\right]} = -\frac{15}{\ln 69} \approx -3.5427.$

So, an equation of the tangent line is

$$y - 2 = -3.5427(x - 13.0279)$$

$$y = -3.5427x + 48.1535.$$

3 pts: {1 pt: considers $\frac{dy/dt}{dx/dt}$; 1 pt: slope at P; 1 pt: equation of tangent line at P}

Reminder: Round each answer to at least three decimal places to receive credit on the exam. Use more than three decimal places when approximating the slope at $t = 3$ in the intermediate step.

(d) $dy/dt = 4t - 3t^2 = 0 \Rightarrow t = 0, 4/3$

$$dx/dt = \ln\left[5 + (1 + t)^3\right] = 0 \Rightarrow t \approx -2.587$$

Because there is no t-value for which both dy/dt and dx/dt equal 0, the particle is never at rest.

2 pts: {1 pt: considers where $dx/dt = 0$ and $dy/dt = 0$; 1 pt: answer with reason}

10. (a) $A = \frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} (2\theta + \cos\theta)^2\, d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} (4\theta^2 + 4\theta\cos\theta + \cos^2\theta)\, d\theta$

$= \frac{1}{2}\left[\frac{4}{3}\theta^3 + 4(\cos\theta + \theta\sin\theta) + \frac{1}{2}(\theta + \sin\theta\cos\theta)\right]_{\pi/2}^{\pi}$

$\approx \frac{1}{2}(38.9125 - 12.2363)$

$= 13.3381$

2 pts: {1 pt: integral; 1 pt: answer}

Reminders: Write down the appropriate definite integral before numerically approximating it on your calculator. No additional work is needed.

Round the answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting the approximation from your calculator. Because this is an approximation, a point may be deducted if an equal sign is used.

(b) $y = r\sin\theta$

$= (2\theta + \cos\theta)\sin\theta$

$= 2\theta\sin\theta + \sin\theta\cos\theta = 1$

$\theta \approx 2.93575$ radians

$x = r\cos\theta$

$= (2\theta + \cos\theta)\cos\theta$

$= 2\theta\cos\theta + \cos^2\theta$

$x(2.93575) = 2(2.93575)\cos(2.93575) + \cos^2(2.93575)$

≈ -4.790

3 pts: {1 pt: finds $y(\theta)$; 1 pt: finds θ for which $y = 1$; 1 pt: finds $x(\theta)$ and computes $x(2.93575)$}

Reminders: Write down the appropriate equation before numerically approximating its solution on your calculator. No additional work is needed.

Round the answer to at least three decimal places to receive credit on the exam. Round θ in the intermediate step to more than three decimal places.

Use "≈" rather than an equal sign in presenting the approximation from your calculator. Because this is an approximation, a point may be deducted if an equal sign is used.

(c) $x = r\cos\theta$

$= (2\theta + \cos\theta)\cos\theta$

$= 2\theta\cos\theta + \cos^2\theta$

$$\frac{dx}{dt} = 2\theta(-\sin\theta)\frac{d\theta}{dt} + 2\frac{d\theta}{dt}\cdot\cos\theta + 2\cos\theta\cdot(-\sin\theta)\frac{d\theta}{dt}$$

When $\theta = \frac{3\pi}{4}$ and $\frac{d\theta}{dt} = 3$,

$$\frac{dx}{dt} = 2\left(\frac{3\pi}{4}\right)\left(-\sin\frac{3\pi}{4}\right)(3) + 2(3)\left(\cos\frac{3\pi}{4}\right) + 2\left(\cos\frac{3\pi}{4}\right)\left(-\sin\frac{3\pi}{4}\right)(3)$$

$\approx -11.239.$

So, the particle is moving to the left on the rectangular coordinate system.

4 pts: {2 pts: Chain Rule with respect to t [computes $(dx/d\theta)\cdot(d\theta/dt)$]; 1 pt: answer; 1 pt: interprets answer}

Reminders: Round the answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting the approximation from your calculator. Because this is an approximation, a point may be deducted if an equal sign is used.

11. (a)

$$A = 3\pi + 2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi} (2 + 2\cos\theta)^2\, d\theta$$

$$= 3\pi + \int_{\pi/3}^{\pi} \left(4 + 8\cos\theta + 4\cos^2\theta\right)^2 d\theta$$

$$= 3\pi + \left[4\theta + 8\sin\theta + 4 \cdot \frac{1}{2}(\theta + \sin\theta\cos\theta)\right]_{\pi/3}^{\pi}$$

$$= 3\pi + \left[(4\pi + 0 + 2\pi) - \left(\frac{4\pi}{3} + 4\sqrt{3} + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$$

$$\approx 14.197$$

3 pts: {1 pt: integrand and constant; 1 pt: limits of integration; 1 pt: answer}

Reminders: Write down the appropriate definite integral before numerically approximating it on your calculator. No further work is needed.

Round the answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting the approximation from your calculator. Because this is an approximation, a point may be deducted if an equal sign is used.

(b)

$$\frac{dr}{dt} = \frac{dr}{d\theta}$$

$$\frac{dr}{dt} \cdot \frac{d\theta}{dr} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dr}$$

$$\frac{d\theta}{dt} = 1$$

$$r = 2 + 2\cos\theta$$

$$\frac{dr}{dt} = -2\sin\theta\,\frac{d\theta}{dt} = -2\sin\theta(1) = -2\sin\theta$$

At $\theta = \frac{\pi}{3}$, $\frac{dr}{dt} = -2\sin\frac{\pi}{3} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} \approx -1.732.$

So, the particle is moving closer to the origin of the polar coordinate system.

3 pts: {1 pt: computes $\frac{dr}{dt} = \frac{d\theta}{dt}$; 1 pt: answer; 1 pt: interpretation}

Reminder: If you choose to give a decimal approximation here, round the answer to at least three decimal places to receive credit on the exam.

(c)

$$\frac{dx}{dt} = \frac{dx}{d\theta}$$

$$\frac{dx}{dt} \cdot \frac{d\theta}{dx} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{d\theta}{dt} = 1$$

$$x = r\cos\theta$$

$$\frac{dx}{dt} = r(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,\frac{dr}{dt}$$

$$= (2 + 2\cos\theta)(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,\frac{dr}{dt}$$

When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dt} = 1$, and $\frac{dr}{dt} = -\sqrt{3}$,

$$\frac{dx}{dt} = \left(2 + 2\cos\frac{\pi}{3}\right)\left(-\sin\frac{\pi}{3}\right)(1) + \cos\left(\frac{\pi}{3}\right)\left(-\sqrt{3}\right)$$

$$= (3)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(-\sqrt{3}\right) = -2\sqrt{3} \approx -3.464$$

So, the particle is moving to the left on the rectangular coordinate system.

3 pts: {1 pt: represents x in terms of θ; 1 pt: computes $\frac{dx}{dt} = \frac{dx}{d\theta}$; 1 pt: answer and interpretation}

12. (a) $A = \frac{1}{2}\int_0^{\pi/6} (1 - 2\sin\theta)^2 \, d\theta$

3 pts: 1 pt: integrand and constant factor; 2 pts: limits of integration

(b) $x = r\cos\theta = (1 - 2\sin\theta)\cos\theta = \cos\theta - 2\sin\theta\cos\theta$

$$\frac{dx}{dt} = -\sin\theta\frac{d\theta}{dt} - 2\left[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)\right]\frac{d\theta}{dt}$$

$$= -\sin\theta\frac{d\theta}{dt} - 2\left(-\sin^2\theta + \cos^2\theta\right)\frac{d\theta}{dt}$$

$$= \left(2\sin^2\theta - 2\cos^2\theta - \sin\theta\right)\frac{d\theta}{dt}$$

$y = r\sin\theta = (1 - 2\sin\theta)(\sin\theta) = \sin\theta - 2\sin^2\theta$

$$\frac{dy}{dt} = (\cos\theta - 4\sin\theta\cos\theta)\frac{d\theta}{dt}$$

3 pts: 1 pt: writes x and y as functions of θ; 1 pt: finds $\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt}$; 1 pt: finds $\frac{dy}{dt} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dt}$

Reminder: You do not need to simplify these derivatives.

(c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 4\sin\theta\cos\theta}{2\sin^2\theta - 2\cos^2\theta - \sin\theta}$

When $\theta = \pi$,

$$\frac{dy}{dx} = \frac{\cos\pi - 4\sin\pi\cos\pi}{2\sin^2\pi - 2\cos^2\pi - \sin\pi}$$

$$= \frac{-1 - 0}{0 - 2(-1)^2 - 0}$$

$$= \frac{1}{2}$$

$x(\theta) = \cos\theta - 2\sin\theta\cos\theta$

$x(\pi) = \cos\pi - 2\sin\pi\cos\pi = -1$

$y(\theta) = \sin\theta - 2\sin^2\theta$

$y(\pi) = \sin\pi = 2\sin^2\pi = 0$

So, an equation of the tangent line is

$$y - 0 = \frac{1}{2}(x + 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

3 pts: 1 pt: writes an expression for dy/dx in terms of θ; 1 pt: finds values for x and y when $\theta = \pi$; 1 pt: equation of tangent line

13. (a) $\frac{dx}{dt}$ is positive at point B because the particle is moving to the right.

$\frac{dy}{dt}$ is positive at point B because the particle is moving upward.

2 pts: 1 pt: answer and explanation for the sign of dx/dt; 1 pt: answer and explanation for the sign of dy/dt

(b) Because there is a cusp at point C, $\frac{dy}{dt}$ is undefined.

$\frac{dy}{dt}$ is undefined when

$$\left(t^3 - \sqrt{27}\right)^{1/3} = 0$$
$$t^3 = 27^{1/2}$$
$$t = 27^{1/6}.$$

So, the particle reaches point C when $t = 27^{1/6} = \sqrt{3}$.

2 pts: 1 pt: reasoning [identifies that dy/dt is undefined at C (cusp)]; 1 pt: answer (finds the t-value where dy/dt is undefined)

(c)
$$y\left(\sqrt{3}\right) = y(0) + \int_0^{\sqrt{3}} y'(t)\, dt$$
$$= 0 - 2\int_0^{\sqrt{3}} \frac{t^2}{\left(t^3 - \sqrt{27}\right)^{1/3}}\, dt \Rightarrow u = t^3 - \sqrt{27},\ du = 3t^2\, dt$$
$$= -2 \cdot \frac{1}{3}\int_0^{\sqrt{3}} \frac{3t^2}{\left(t^3 - \sqrt{27}\right)^{1/3}}\, dt$$
$$= -\frac{2}{3}\left[\frac{3}{2}\left(t^3 - \sqrt{27}\right)^{2/3}\right]_0^{\sqrt{3}}$$
$$= -\left[\left(\left(\sqrt{3}\right)^3 - \sqrt{27}\right)^{2/3} - \left(0 - \sqrt{27}\right)^{2/3}\right]$$
$$= 3$$

3 pts: 1 pt: antiderivative (shows substitution); 1 pt: uses initial condition, $y(0) = 0$; 1 pt: answer

(d)
$$y = -\frac{2}{3}x + \frac{20}{3}$$
$$\frac{dy}{dt} = -\frac{2}{3}\frac{dx}{dt}$$

Because $\frac{dx}{dt} \approx 1.837$, $\frac{dy}{dt} = -\frac{2}{3}(1.837)$.

So, the speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1.837)^2 + \left[-\frac{2}{3}(1.837)\right]^2}$.

2 pts: 1 pt: finds dy/dt at this point [Chain Rule: $dy/dt = (dy/dx) \cdot (dx/dt)$]; 1 pt: finds the speed

Reminder: These approximations do not need to be simplified.