AP® Exam Practice Questions for Chapter 5

1. To find which graph is a slope field for $\frac{dy}{dx} = y - \frac{x}{5}$, evaluate the derivative at selected points.

At $(0, 1)$, $\frac{dy}{dx} = 1$.

At $(1, 0)$, $\frac{dy}{dx} = -\frac{1}{5}$.

At $(5, 0)$, $\frac{dy}{dx} = -1$,

So, the answer is B.

2.
$$\frac{dP}{dt} = kP$$
$$\int \frac{1}{P}\,dP = \int k\,dt$$
$$\ln|P| = kt + C_1$$
$$P = Ce^{kt}$$

Use $(15, 3)$ and $C = 1$ to find k.

$$3 = 1e^{15k}$$
$$\ln 3 = 15k$$
$$\frac{\ln 3}{15} = k$$

So, the answer is B.

3.
$$y' = ky$$
$$\frac{1}{y}y' = k$$
$$\int \frac{1}{y}\,dy = \int k\,dx$$
$$\ln|y| = kx + C_1$$
$$y = Ce^{kx}$$

Use $f(0) = 8$ and $f(6) = 2$ to find C and k.

$$8 = Ce^{k(0)} \Rightarrow C = 8$$
$$2 = Ce^{k(6)}$$
$$2 = 8e^{6k}$$
$$\frac{1}{4} = e^{6k}$$
$$\frac{1}{6}\ln\frac{1}{4} = k$$

Because $f(x) = Ce^{kx} = 8e^{(x/6)\ln(1/4)}$, the answer is A.

4.
$$\frac{dy}{dx} = 2xy^2$$
$$\frac{1}{y^2}\,dy = 2x\,dx$$
$$\int \frac{1}{y^2}\,dy = \int 2x\,dx$$
$$-\frac{1}{y} = x^2 + C$$

Use $y(-1) = 2$ to find C.

$$-\frac{1}{2} = (-1)^2 + C \Rightarrow C = -\frac{3}{2}$$
$$-\frac{1}{y} = x^2 - \frac{3}{2} \Rightarrow y = -\frac{2}{2x^2 + 3}$$
$$y(2) = -\frac{2}{2(2)^2 - 3} = -\frac{2}{5}$$

So, the answer is C.

5.
$$\frac{dy}{dx} = \frac{3y}{x}$$
$$\frac{1}{y}\,dy = \frac{3}{x}\,dx$$
$$\int \frac{1}{y}\,dy = 3\int \frac{1}{x}\,dx$$
$$\ln|y| = 3\ln|x| + C_1$$
$$\ln|y| = \ln|x^3| + C_1$$
$$y = Cx^3$$

Use $y(1) = -1$ to find C.

$$-1 = C(1)^3$$
$$-1 = C$$

The solution of the differential equation is $y = -x^3$.

So, the answer is B.

6. Evaluate each differential equation for selected values of y.

A: When $y = 2, \dfrac{dy}{dx} = \dfrac{20}{3}$.

When $y = 3, \dfrac{dy}{dx} = 0$.

When $y = 4, \dfrac{dy}{dx} = -\dfrac{40}{3}$.

B: When $y = 2, \dfrac{dy}{dx} = \dfrac{1}{3}$.

When $y = 3, \dfrac{dy}{dx} = 0$.

When $y = 4, \dfrac{dy}{dx} = -\dfrac{2}{3}$.

C: When $y = 2, \dfrac{dy}{dx} = \dfrac{2}{3}$.

When $y = 3, \dfrac{dy}{dx} = 0$.

When $y = 4, \dfrac{dy}{dx} = -\dfrac{4}{3}$,

D: When $y = 2, \dfrac{dy}{dx} = \dfrac{20}{3}$.

When $y = 3, \dfrac{dy}{dx} = \dfrac{15}{2}$.

When $y = 4, \dfrac{dy}{dx} = \dfrac{20}{3}$.

So, the answer is A.

7. Use Euler's Method with $y(0) = -3$, and $h = \frac{1}{3}$ to find $y(1)$.

$y\left(\frac{1}{3}\right) \approx -3 + \frac{1}{3}(-2) = -\frac{11}{3}$

$y\left(\frac{2}{3}\right) \approx -\frac{11}{3} + \frac{1}{3}\left(-\frac{28}{9}\right) = -\frac{127}{27}$

$y(1) \approx -\frac{127}{27} + \frac{1}{3}(-4.753) \approx -6.288$

So, the answer is A.

8. (a)

$$\frac{dy}{dt} = 0.5y$$

$$\frac{1}{y}\,dy = 0.5\,dt$$

$$\int \frac{1}{y}\,dy = \int 0.5\,dt$$

$$\ln|y| = 0.5t + C_1$$

$$y = Ce^{0.5t}$$

Use $(0, 200)$ to find C.

$$200 = Ce^{0.5(0)} \Rightarrow C = 200$$

So, $y = 200e^{0.5t}$.

6 pts:
- 1 pt: separation of variables
- 2 pts: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes:

3 points max if no constant of integration present

0 points if no separation of variables

(b)

$$\frac{1}{10}\int_0^{10} 200e^{0.5t}\,dt = \frac{1}{10}\cdot 2\int_0^{10} 200e^{0.5t}(0.5)\,dt$$

$$= \frac{1}{5}\Big[200e^{0.5t}\Big]_0^{10}$$

$$= 40(e^5 - 1)$$

$$\approx 5896.526 \text{ bacteria}$$

3 pts:
- 2 pts: sets up average value definite integral
- 1 pt: answer with units

Reminders:

Write down the appropriate definite integral before numerically approximating it on your calculator.

Be sure to round the answer to at least three decimal places to receive credit.

Note:

Importing an incorrect particular solution from part (a) can still earn the first two points in part (b) (but not the answer point).

9. (a) $\frac{dy}{dx} = \frac{1}{xy}$

$$\frac{d^2y}{dx^2} = \frac{xy(0) - \left[x\frac{dy}{dx} + y\right]}{(xy)^2}$$

$$= -\frac{x\left(\frac{1}{xy}\right) + y}{x^2y^2}$$

$$= -\frac{\frac{1}{y} + y}{x^2y^2}$$

$$= -\frac{1 + y^2}{x^2y^3}$$

At the point $(1, 2)$, $\frac{d^2y}{dx^2} = -\frac{1 + (2)^2}{(1)^2(2)^3} = -\frac{5}{8}$.

2 pts: implicit differentiation to find d^2y/dx^2 (in terms of x and y); evaluates

(b) Find $\frac{dy}{dx}$ at the point $(1, 2)$.

$$\frac{dy}{dx} = \frac{1}{xy} = \frac{1}{(1)(2)} = \frac{1}{2}$$

So, the equation of the tangent line is

$$y - 2 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{3}{2}.$$

$$f(1.1) \approx \frac{1}{2}(1.1) + \frac{3}{2} = 2.05$$

Because $f''(1.1) < 0$, the approximation for $f(1.1) \approx 2.05$ is greater than $f(1.1)$.

2 pts:
- 1 pt: finds equation of tangent line using dy/dx at $(1, 2)$ and approximates $f(1.1)$
- 1 pt: answer ("greater than") with reason (appeals to the concavity of f)

(c) $\frac{dy}{dx} = \frac{1}{xy}$

$$y\,dy = \frac{1}{x}\,dx$$

$$\int y\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}y^2 = \ln|x| + C$$

Use $f(1) = 2$ to find C.

$$\frac{1}{2}(2) = \ln(1) + C$$

$$2 = 0 + C$$

$$2 = C$$

So, the solution is

$$\frac{1}{2}y^2 = \ln|x| + 2$$

$$y^2 = 2\ln|x| + 4$$

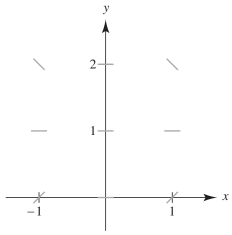
$$y = \sqrt{2\ln|x| + 4}.$$

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

10. (a)



1 pt: correct slopes at points where $x = 0$ or $y = 1$ (horizontal line segments)

(b) $\frac{dy}{dx} = x^2(1 - y) > 0$

$$1 - y > 0$$
$$-y > -1$$
$$y < 1$$

So, the slopes are positive when $y < 1$.

1 pt: answer with justification

(c)
$$\frac{dy}{dx} = x^2(1 - y)$$
$$\frac{1}{y - 1}\,dy = -x^2\,dx$$
$$\int \frac{1}{y - 1}\,dy = -\int x^2\,dx$$
$$\ln\left|y - 1\right| = -\frac{1}{3}x^3 + C_1$$
$$y - 1 = e^{(-1/3)x^3}e^{C_1}$$
$$y = 1 + Ce^{(-1/3)x^3}$$

Use $f(0) = 2$ to find C.

$$2 = 1 + Ce^{(-1/3)(0)^3}$$
$$2 = 1 + C$$
$$C = 1$$

So, the solution is $f(x) = 1 + e^{(-1/3)x^3}$.

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes:

2 points max if no constant of integration present

0 points if no separation of variables

(d) $\frac{dy}{dx} = x^2(1 - y)$

$$\frac{d^2y}{dx^2} = 2x(1 - y) + x^2\left(-\frac{dy}{dx}\right)$$
$$= 2x(1 - y) - x^2\left[x^2(1 - y)\right]$$
$$= x(1 - y)\left(2 - x^3\right)$$

$f(x) = 1 + e^{-x^3/3}$

$f'(x) = -x^2\,e^{-x^3/3} = 0$ when $x = 0$.

Interval:	$(-\infty, 0)$	$(0, \infty)$
Sign of f':	$f' < 0$	$f' < 0$
Conclusion:	Decreasing	Decreasing

So, f has neither a relative minimum nor maximum at $x = 0$.

2 pts:
- 1 pt: finds $\frac{d^2y}{dx^2}$
- 1 pt: answer with reason (applies the First Derivative Test to determine)

11. (a) Use Euler's Method with $y' = \frac{2x}{y}, f(1) = 2,$ and $h = 0.2$ to approximate $f(1.4)$.

$$y_1 = y_0 + hF(x_0, y_0) = 2 + 0.2\left[\frac{2(1)}{2}\right] = 2.2$$

$$y_2 = y_1 + hF(x_1, y_1) = 2.2 + 0.2\left[\frac{2(1.2)}{2.2}\right]$$

So, $f(1.4) \approx 2.2 + 0.2\left[\frac{2.4}{2.2}\right]$.

3 pts: $\begin{cases} \text{2 pts: Euler's Method with two steps} \\ \text{1 pt: answer [approximation of } f(1.4)] \end{cases}$

Reminders: The answer does *not* need to be simplified. Leaving the answer as $2.2 + 0.2\left[\frac{2(1.2)}{2.2}\right]$ is recommended.

Be sure to write "$f(1.4) \approx$" rather than "$f(1.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used. In general, equating two quantities that are not truly equal will result in a one point deduction on a free-response question.

(b) $$\frac{dy}{dx} = \frac{2x}{y}$$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 = 2x^2 + C$$

Use $(1, 2)$ to find C.

$$(2)^2 = 2(1)^2 + C$$

$$4 = 2 + C$$

$$C = 2$$

Because $y^2 = 2x^2 + 2$, the solution is $y = \sqrt{2x^2 + 2}$, where the domain is $(-\infty, \infty)$.

6 pts: $\begin{cases} \text{1 pt: separation of variables} \\ \text{1 pt: antiderivatives} \\ \text{1 pt: constant of integration} \\ \text{1 pt: uses initial condition} \\ \text{1 pt: particular solution (solves for } y) \\ \text{1 pt: states domain of } y \end{cases}$

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

12. (a) Use Euler's Method, $\dfrac{dy}{dx} = xy,\ f(1) = 1,$ and $h = 0$ to find $f(1.2)$.

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1[(1)(1)] = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1[(1.1)(1.1)]$$

So, $f(1.2) \approx 1.1 + 0.1[(1.1)(1.1)]$.

2 pts: $\begin{cases}\text{1 pt: Euler's Method with two steps}\\ \text{1 pt: answer } [\text{approximation of } f(1.2)]\end{cases}$

Reminders: The answer does *not* need to be simplified. Leaving the answer as $1.1 + 0.1[(1.1)(1.1)]$ is recommended.

Be sure to write "$f(1.2) \approx$" rather than "$f(1.2) =$." Because this is an approximation, a point may be deducted if an equal sign is used. In general, equating two quantities that are not truly equal will result in a one point deduction on a free-response question.

(b) $\dfrac{dy}{dx} = xy$

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y(1)$$

$$= x(xy) + y$$

$$= x^2y + y$$

On the interval $[1, 1.2]$, x is positive. Because $\dfrac{dy}{dx} = xy$, $\dfrac{dy}{dx}$ and y have the same sign on $[1, 1.2]$. Because $y = f(1) = 1$, $\dfrac{dy}{dx}$ and y are both positive on $[1, 1.2]$. Therefore, $\dfrac{d^2y}{dx^2} > 0$, so the solution y is concave upward on this interval and any tangent line approximation will be *below* the curve y. So, the approximation found in part (a) is less than $f(1.2)$.

2 pts: $\begin{cases}\text{1 pt: implicit differentiation to find } d^2y/dx^2 \text{ (in terms of } x \text{ and } y)\\ \text{1 pt: answer with reason (appeals to the concavity of } y \text{ on this interval)}\end{cases}$

(c) $\dfrac{dy}{dx} = xy$

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C_1$$

$$y = e^{(1/2)x^2}e^{C_1}$$

$$y = Ce^{(1/2)x^2}$$

Use $(1, 1)$ to find C.

$$1 = Ce^{(1/2)(1)^2}$$

$$1 = Ce^{1/2} \Rightarrow C = e^{-1/2}$$

So, the solution is

$$y = e^{-1/2}e^{(1/2)x^2} = e^{(1/2)x^2 - 1/2} = e^{0.5x^2 - 0.5}.$$

5 pts: $\begin{cases}\text{1 pt: separation of variables}\\ \text{1 pt: antiderivatives}\\ \text{1 pt: constant of integration}\\ \text{1 pt: uses initial condition}\\ \text{1 pt: particular solution (solves for } y)\end{cases}$

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

13. (a) At $y = 100$, $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$

$$= \frac{1}{10}(100)\left(1 - \frac{100}{1000}\right)$$

$$= 10\left(\frac{9}{10}\right)$$

$$= 9.$$

At $y = 200$, $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$

$$= \frac{1}{10}(200)\left(1 - \frac{200}{1000}\right)$$

$$= 20\left(\frac{8}{10}\right)$$

$$= 16.$$

Because $\frac{dy}{dt} = 9$ when $y = 100$ is less than $\frac{dy}{dt} = 16$ when $y = 200$, the disease is spreading faster when 200 people have the disease.

3 pts: {2 pts: Computes dy/dt at $y = 100$ and $y = 200$; 1 pt: conclusion with reason}

Reminder: To avoid the risk of an arithmetic mistake, these answers do *not* need to be simplified.

(b) $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$, where $L = 1000$ and $k = \frac{1}{10}$.

So, $y = \frac{1000}{1 + Ce^{(-1/10)t}}$.

Use $(0, 100)$ to find C.

$$100 = \frac{1000}{1 + Ce^{(-1/10)(0)}}$$

$$1 + C = 10$$

$$C = 9$$

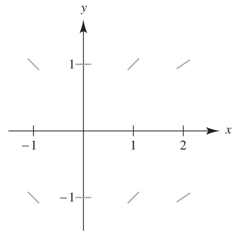
So, a model for the population is $y = \frac{1000}{1 + 9e^{(-1/10)t}}$.

5 pts: {2 pts: general solution of the logistic differential equation; 1 pt: constant of integration; 1 pt: uses initial condition; 1 pt: particular solution}

(c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1000}{1 + 9e^{(-1/10)t}} = \frac{1000}{1 + 0} = 1000$

1 pt: answer

14. (a)



2 pts: slopes of line segments

(b) $\dfrac{dy}{dx} = \dfrac{x}{y^2} = xy^{-2}$

$$\frac{d^2y}{dx^2} = x\left[-2y^{-3}\frac{dy}{dx}\right] + y^{-2}(1)$$

$$= -\frac{2x}{y^3}\left(\frac{x}{y^2}\right) + \frac{1}{y^2}$$

$$= \frac{-2x^2 + y^3}{y^5}$$

2 pts: implicit differentiation to find d^2y/dx^2 (in terms of x and y)

(c) $\dfrac{dy}{dx} = \dfrac{x}{y^2}$

$$y^2\,dy = x\,dx$$

$$\int y^2\,dy = \int x\,dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

Use $y(0) = 2$ to find C.

$$\frac{1}{3}(2)^3 = \frac{1}{2}(0)^2 + C \Rightarrow C = \frac{8}{3}$$

So, the solution is

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + \frac{8}{3}$$

$$y^3 = \frac{3}{2}x^2 + 8 \Rightarrow y = \sqrt[3]{\frac{3}{2}x^2 + 8}.$$

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables