AP® Practice Exam: Calculus AB

Section 1

1. $g(x) = 2\cot 3x$

$$g'(x) = 2\lim_{h\to 0}\frac{\cot(3(x+h)) - \cot 3x}{h}$$

$$g'\left(\frac{\pi}{6}\right) = 2\lim_{h\to 0}\frac{\cot\left(3\left(\frac{\pi}{6}+h\right)\right) - \cot\left(3\cdot\frac{\pi}{6}\right)}{h}$$

$$= 2\lim_{h\to 0}\frac{\cot\left(\frac{\pi}{2}+3h\right) - \cot\left(\frac{\pi}{2}\right)}{h}$$

So, the answer is A.

2. $\frac{x+x^3}{1+x} = x^2 - x + 2 - \frac{2}{x+1}$ (by long division)

$$6\int_0^1 \frac{x+x^3}{1+x}\,dx = 6\int_0^1\left(x^2 - x + 2 - \frac{2}{x+1}\right)dx$$

$$= 6\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x - 2\ln|x+1|\right]_0^1$$

$$= 6\left(\frac{1}{3} - \frac{1}{2} + 2 - 2\ln 2\right)$$

$$= 11 - 12\ln 2$$

So, the answer is B.

3. $f(x) = \sin\left(\pi e^{-3x}\right)$

$$f'(x) = \cos\left(\pi e^{-3x}\right)\cdot -3\pi e^{-3x}$$

$$f'(0) = \cos(\pi)\cdot -3\pi$$

$$= 3\pi$$

So, the answer is D.

4. $g(-4) = 0$

Because g is decreasing at $x = -4$, $g'(-4) < 0$.

Because g is concave upward at $x = -4$, $g''(-4) > 0$.

So, $g'(-4) < g(-4) < g''(-4)$.

So, the answer is B.

5. $\frac{f(4) - f(0)}{4 - 0} = \frac{27 - 3}{4} = 6$

The Mean Value Theorem guarantees that $f'(c) = 6$ for at least one value of c in $[0, 4]$.

So, the answer is D.

6. $\int_{-1}^{2} h(x)\,dx = \int_{-1}^{0} e^{3x}\,dx + \int_0^2 \cos\frac{x}{5}\,dx$

$$= \left[\frac{1}{3}e^{3x}\right]_{-1}^{0} + \left[5\sin\frac{x}{5}\right]_0^2$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3} + 5\sin\frac{2}{5}$$

$$= \frac{e^3 - 1}{3e^3} + 5\sin\frac{2}{5}$$

So, the answer is D.

7. $g(x) = \sqrt[3]{27 - x^3}$

$$g(g(x)) = \sqrt[3]{27 - \left(\sqrt[3]{27 - x^3}\right)^3}$$

$$= \sqrt[3]{27 - (27 - x^3)}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

$$\frac{d}{dx}\left[g(g(x))\right] = 1$$

At $x = -8$, $\frac{d}{dx}\left[g(g(x))\right] = 1$.

So, the answer is C.

8. $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 2x \csc 7x = \lim_{x \to 0^-} \frac{2x}{\sin 7x}$

Because an indeterminate form $\frac{0}{0}$ is achieved,

L'Hôpital's Rule applies:

$\lim_{x \to 0^-} \frac{2x}{\sin 7x} = \lim_{x \to 0^-} \frac{2}{7 \cos 7x} = \frac{2}{7}.$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} k\left[1 + \ln\left(x + e^{x+1}\right)\right]$$

$$= k[1 + \ln e] = 2k$$

For g to be continuous at $x = 0$,

$\lim_{x \to 0} g(x)$ must exist $\Rightarrow \frac{2}{7} = 2k \Rightarrow k = \frac{1}{7}.$

So, the answer is C.

9. $y = \frac{\sqrt{x+2}}{2x - 6}$

$$y' = \frac{(2x-6)\frac{1}{2\sqrt{x+2}} - \sqrt{x+2}(2)}{(2x-6)^2}$$

$$y'(2) = \frac{(-2)\left(\frac{1}{4}\right) - (2)(2)}{(-2)^2} = \frac{-\frac{1}{2} - 4}{4} = -\frac{9}{8}$$

An equation of the line tangent to y at $(2, -1)$ is

$$y + 1 = -\frac{9}{8}(x - 2)$$

$$8y + 8 = -9x + 18$$

$$9x + 8y = 10.$$

So, the answer is B.

10. $$\int \csc^2 x \cos x(1 + \cos x)\, dx = \int \left(\csc^2 x \cos x + \csc^2 x \cos^2 x\right) dx$$

$$= \int \left(\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} + \frac{\cos^2 x}{\sin^2 x}\right) dx$$

$$= \int \left(\cot x \csc x + \cot^2 x\right) dx$$

$$= \int \left(\cot x \csc x + \csc^2 x - 1\right) dx$$

$$= -\csc x - \cot x - x + C$$

So, the answer is A.

11. $f(x) = 5 + x - 2x^2 + x^3$

$f'(x) = 1 - 4x + 3x^2$

$0 = 1 - 4x + 3x^2$

$0 = (1 - 3x)(1 - x)$

f has critical numbers at $x = \frac{1}{3}$ and $x = 1$.

Interval:	$-\infty < x < \frac{1}{3}$	$\frac{1}{3} < x < 1$	$1 < x < \infty$
Test Value:	0	$\frac{2}{3}$	2
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	*f* increasing	*f* decreasing	*f* increasing

$f''(x) = -4 + 6x$

$0 = -4 + 6x$

$\frac{2}{3} = x$

Interval:	$-\infty < x < \frac{2}{3}$	$\frac{2}{3} < x < \infty$
Test Value:	0	1
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	*f* is concave downward	*f* is concave upward

f is both increasing and concave upward when $x > 1$.

So, the answer is D.

12. $\frac{dy}{dt} = \frac{y^2}{2t}$

$\frac{dy}{y^2} = \frac{1}{2t}\,dt$

$\frac{y^{-1}}{-1} = \frac{1}{2}\ln|t| + C$

$\frac{1}{y} = -\ln\sqrt{t} + C$

$y = -\frac{1}{\ln\sqrt{t} + C}$

So, the answer is A.

13. $f(x) = \ln\left[(2 - x)^4\right] = 4\ln(2 - x)$

$f'(x) = -\frac{4}{2 - x}$

f is not continuous at $x = 2$ because $f(2)$ is undefined.

Therefore, *f* is not differentiable at $x = 2$.

So, the answer is D.

14. $f(x) = e^{-0.8x^2}$

$A(x) = xe^{-0.8x^2}$

$A'(x) = -1.6x^2e^{-0.8x^2} + e^{-0.8x^2}$

$0 = e^{-0.8x^2}\left(-1.6x^2 + 1\right)$

$0 = -1.6x^2 + 1 \Rightarrow x^2 = \frac{1}{1.6} \Rightarrow x = \frac{1}{\sqrt{1.6}} = \frac{\sqrt{1.6}}{1.6}$

Interval:	$0 < x < \frac{\sqrt{1.6}}{1.6}$	$\frac{\sqrt{1.6}}{1.6} < x < \infty$
Test Value:	$\frac{1}{2}$	1
Sign of A':	$A' > 0$	$A' < 0$

A has a relative maximum at $x = \frac{\sqrt{1.6}}{1.6}$ because A' changes from positive to negative at $x = \frac{\sqrt{1.6}}{1.6}$.

So, the answer is A.

15. $\frac{dy}{dx} = \frac{y \ln x^2}{x}, y(1) = 5$

$\frac{dy}{y} = \frac{2 \ln x}{x}\,dx$

$\ln|y| = (\ln x)^2 + C_1$

$y = e^{(\ln x)^2} \cdot e^{C_1}$

$y = C_2\left(e^{\ln x}\right)^{\ln x}$

$y = C_2\, x^{\ln x}$

$5 = C_2(1)^{\ln 1} \Rightarrow 5 = C_2$

$y = 5x^{\ln x}$

So, the answer is C.

16.
$$\lim_{x\to 1}\frac{\left(x^2 + x - 2\right)f(x)}{x - 1} = \lim_{x\to 1}\frac{(x + 2)(x - 1)f(x)}{x - 1}$$
$$= \lim_{x\to 1}(x + 2)f(x)$$
$$= \lim_{x\to 1}(x + 2) \cdot \lim_{x\to 1} f(x)$$
$$= (3)(4)$$
$$= 12$$

So, the answer is C.

17. $x^2 - 2y^2 = 2$

$2x - 4y\frac{dy}{dx} = 0$

$-4y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{x}{2y}$

At $(2, 1)$, $\frac{dy}{dx} = \frac{2}{2} = 1.$

$\frac{d^2y}{dx^2} = \frac{2y - 2x\frac{dy}{dx}}{4y^2}$

At $(2, 1)$, $\frac{d^2y}{dx^2} = \frac{(2)(1) - (2)(2)(1)}{4(1)^2} = \frac{2 - 4}{4} = -\frac{1}{2}.$

So, the answer is B.

18. $g = e^{kx^4}$

$g' = 4kx^3e^{kx^4}$

$g'' = 16k^2x^6e^{kx^4} + 12kx^2e^{kx^4}$

$= 4kx^2e^{kx^4}\left(4kx^4 + 3\right)$

When $x = \pm 1$, $g'' = 4ke^k(4k + 3)$.

$g'' = 0$ when $k = 0$ or $k = -\frac{3}{4}$.

So, the answer is A.

19. $2 - x^2 = x - 10$

$0 = x^2 + x - 12$

$0 = (x + 4)(x - 3)$

The graphs intersect at $x = -4$ and $x = 3$.

$$A = \int_{-4}^{3} \left[\left(2 - x^2\right) - (x - 10)\right] dx$$

$$= \int_{-4}^{3} \left(12 - x - x^2\right) dx$$

$$= \left[12x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-4}^{3}$$

$$= \left(36 - \frac{9}{2} - 9\right) - \left(-48 - 8 + \frac{64}{3}\right)$$

$$= \frac{343}{6}$$

So, the answer is B.

20. $y = \dfrac{e^{\sqrt{2x}}}{\ln 3x}$

$$\frac{dy}{dx} = \frac{(\ln 3x)\left(\dfrac{e^{\sqrt{2x}}}{\sqrt{2x}}\right) - \left(e^{\sqrt{2x}}\right)\left(\dfrac{1}{x}\right)}{\ln^2 3x}$$

$$= \frac{e^{\sqrt{2x}}\left(x \ln 3x - \sqrt{2x}\right)}{x\sqrt{2x}\ \ln^2 3x}$$

So, the answer is B.

21. $p(x) = (x + 2)^2(x - 1)$

$p'(x) = (x + 2)^2 + (x - 1) \cdot 2(x + 2)$

$0 = (x + 2)\left[(x + 2) + 2(x - 1)\right]$

$0 = (x + 2)(3x)$

$x = -2,\ x = 0$

$p''(x) = 6x + 6$

$p''(0) = 6 > 0 \Rightarrow p$ has a relative minimum at $x = 0$.

So, the answer is C.

22. In the given slope field, the slopes are positive at all points $(t, y) \Rightarrow \dfrac{dy}{dt} > 0$ for all (t, y).

Because $\dfrac{dy}{dt} = y^2 > 0$ for all (t, y), the answer is B.

23. $y = \sqrt{1 + e^{\sec x} + \tan x^2}$

$$y' = \frac{1}{2}\left(1 + e^{\sec x} + \tan x^2\right)^{-1/2}\left(e^{\sec x} \sec x \tan x + 2x \sec^2 x^2\right)$$

$$= \frac{e^{\sec x} \sec x \tan x + 2x \sec^2 x^2}{2\sqrt{1 + e^{\sec x} + \tan x^2}}$$

So, the answer is D.

24. $\lim_{x\to 0} \dfrac{2x - \sin 2x}{3x(1 - \cos x)}$

Because an indeterminate form $\frac{0}{0}$ is achieved, L'Hôpital's Rule applies.

$$\lim_{x\to 0} \frac{2x - \sin 2x}{3x(1 - \cos x)} = \lim_{x\to 0} \frac{2 - 2\cos 2x}{3x \sin x + 3(1 - \cos x)}$$

Because an indeterminate form $\frac{0}{0}$ is achieved, L'Hôpital's Rule applies.

$$\lim_{x\to 0} \frac{2 - 2\cos 2x}{3x \sin x + 3(1 - \cos x)} = \lim_{x\to 0} \frac{4 \sin 2x}{3x\cos x + 3\sin x + 3\sin x}$$

Because an indeterminate form $\frac{0}{0}$ is achieved, L'Hôpital's Rule applies.

$$\lim_{x\to 0} \frac{4\sin 2x}{3x\cos x + 6\sin x} = \lim_{x\to 0} \frac{8\cos 2x}{-3x\sin x + 3\cos x + 6\cos x}$$
$$= \frac{8}{3 + 6}$$
$$= \frac{8}{9}$$

So, the answer is C.

25. $\lim_{x\to\infty} \dfrac{6x^4 + x^2 - 1}{3x^4 + 7x^3 + 2x} = \lim_{x\to\infty} \dfrac{6 + \frac{1}{x^2} - \frac{1}{x^4}}{3 + \frac{7}{x} + \frac{2}{x^3}} = \dfrac{6}{3} = 2$

So, the answer is B.

26.
$$e^x = 4 - e^{-x}$$
$$e^{2x} = 4e^x - 1$$
$$e^{2x} - 4e^x + 1 = 0$$
$$e^x = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$
$$x = \ln\left(2 \pm \sqrt{3}\right)$$

So, the intersection points occur at $x = \ln\left(2 \pm \sqrt{3}\right)$.

$R(x) = 4 - e^{-x}, r(x) = e^x$

$$V = \pi\int_{\ln(2-\sqrt{3})}^{\ln(2+\sqrt{3})} \left[\left(4 - e^{-x}\right)^2 - \left(e^x\right)^2\right] dx$$

So, the answer is C.

27. $\dfrac{1}{30}\left(\ln\dfrac{31}{30} + \ln\dfrac{32}{30} + \ln\dfrac{33}{30} + \cdots + \ln 3\right) = \dfrac{1}{30}\sum_{i=1}^{60} \ln\left(\dfrac{30 + i}{30}\right)$

$$= \frac{2}{60}\sum_{i=1}^{60} \ln\left(\frac{30 + i}{30}\right)$$
$$\approx \int_1^3 \ln x\, dx$$

So, the answer is D.

28. $\frac{d}{dt}\left(\sqrt{t^3}\int_2^t \cot(u+1)\,du\right)$

$$= \sqrt{t^3}\cot(t+1) + \frac{3}{2}\sqrt{t}\int_2^t \cot(u+1)\,du$$

So, the answer is A.

29. $v(t) = \ln(t^2+2) - 3$

$v(t) < 0$ on $[0, 4.2527]$

$v(t) > 0$ on $[4.2527, 8]$

$$\text{Distance} = \int_0^8 |v(t)|\,dt$$
$$= -\int_0^{4.2527} v(t)\,dt + \int_{4.2527}^8 v(t)\,dt$$
$$\approx 7.406$$

So, the answer is A.

30. $f(x) = e^{x+2},\ g(x) = 3x - x^2$

$f'(x) = e^{x+2},\ g'(x) = 3 - 2x$

Find where $f'(x) = -\frac{1}{g'(x)}$.

$$e^{x+2} = -\frac{1}{3-2x}$$
$$x \approx 1.515$$

So, the answer is D.

31. $f(x) = 2 - x^2 + \int_0^x (t-3)\sin\frac{\pi t}{3}\,dt$

$f'(x) = -2x + (x-3)\sin\frac{\pi x}{3}$

$f''(x) = -2 + \frac{\pi}{3}(x-3)\cos\frac{\pi x}{3} + \sin\frac{\pi x}{3}$

Because $f(0) = 2$ and f is continuous on $(0, 3)$, $f(x)$ is not less than 0 on the interval $0 < x < 3$. From the graphs of $f'(x)$ and $f''(x)$, we see that $f'(x) < 0$ and $f''(x) < 0$ on the interval $0 < x < 3$.

So, the answer is D.

32. $g(x) = \int_0^x \left(4f(t) - \sqrt{t^5+4}\right)dt,\ f(2) = 3,\ \int_0^2 f(t)\,dt = 13$

$g(2) = 4\int_0^2 f(t)\,dt - \int_0^2 \sqrt{t^5+4}\,dt$

$\approx 52 - 5.6856 \approx 46.3144$

$g'(x) = 4f(x) - \sqrt{x^5+4}$

$g'(2) = 4f(2) - \sqrt{2^5+4} = 12 - 6 = 6$

$g(2) + 2g'(2) = 46.3144 + 12 \approx 58.314$

So, the answer is D.

33. $v(t) = \frac{1}{t-2}\cos\left(2\pi\sqrt{t}\right)$

$v(t) < 0$ on $(0, 0.0625)$ and $(0.5625, 1)$

$v(t) > 0$ on $(0.0625, 0.5625)$

$$\text{Total distance} = \int_0^1 |v(t)|\,dt$$
$$= -\int_0^{0.0625} v(t)\,dt + \int_{0.0625}^{0.5625} v(t)\,dt - \int_{0.5625}^1 v(t)\,dt$$
$$\approx 0.0146 + 0.1864 + 0.2509$$
$$\approx 0.452$$

So, the answer is C.

34. $\int_{0.1}^{2.9} h(x)\,dx \approx \frac{2.8}{8}\left[h(0.1) + 2h(0.8) + 2h(1.5) + 2h(2.2) + h(2.9)\right]$

$$\approx 0.35\left[0.001 + 2(0.178) + 2(0.882) + 2(1.008) + 2.073\right]$$
$$\approx 2.174$$

So, the answer is B.

35. $g(x) = \ln 2x^2 + \sqrt[5]{x^4 - 15},\ x > 0$

$5 = \ln 2x^2 + \sqrt[5]{x^4 - 15}$

$x \approx 2.8614$

$g'(2.8614) \approx 1.4929$

So, $h'(5) \approx \dfrac{1}{g'(2.8614)} \approx \dfrac{1}{1.4929} \approx 0.6698$

So, the answer is B.

36. $d(x) = \sqrt{24 - e^x},\ 0 \le x \le 3$

$$\text{Average density} = \frac{1}{3}\int_0^3 \sqrt{24 - e^x}\,dx \approx 4.142$$

So, the answer is B.

37. $f(x) = \ln(x^2)e^{-1/x},\ y = 2 - 2x$

$\ln(x^2)e^{-1/x} = 2 - 2x$

$x = 1$

$f'(1) = 0.73576$

So, the slope of the line normal to f at this point is

$m \approx -\dfrac{1}{0.73576} \approx -1.3591.$

So, the answer is B.

38. $\displaystyle\int_0^7 \frac{3t^2}{2(t^3 + 1)}\,dt \approx 2.920$

The cat gained about 2.920 pounds.

So, the answer is A.

39. $x(t) = \sin(2t^2)$

$v(5) = x'(5) \approx 19.299$

So, the answer is C.

40. $f(-1) = 3,\ f'(x) \le 7$

The greatest possible value for $f(5)$ would be $3 + 7(6) = 45$. (Note: An equation of the line through $(-1, 3)$ with slope 7 is $y = 7x + 10$.)

So, the answer is D.

41. $r(x) = \frac{1}{2}(1 - \cos^2 x - \cos^2 x) = \frac{1}{2}(1 - 2\cos^2 x)$

$$\begin{aligned} V &= \int_{\pi/4}^{3\pi/4} A(x)\,dx \\ &= \int_{\pi/4}^{3\pi/4} \frac{1}{2}\pi\left[r(x)\right]^2 dx \\ &= \frac{\pi}{8}\int_{\pi/4}^{3\pi/4} (1 - 2\cos^2 x)^2\,dx \\ &= \frac{\pi}{8}\left(\frac{\pi}{4}\right) \\ &\approx 0.308 \end{aligned}$$

So, the answer is A.

42. $g(x) = \dfrac{e^x + x^2}{x + 2}$

$$g'(x) = \frac{(x + 2)(e^x + 2x) - (e^x + x^2)}{(x + 2)^2} = \frac{xe^x + x^2 + e^x + 4x}{(x + 2)^2}$$

$g'(x) = 0$ when $x \approx -0.1795$ and $x \approx -4.0136$.

So, the answer is A.

43. $A = 2(1 + \sqrt{2})s^2,\ P = 8s \Rightarrow \dfrac{P}{8} = s,\ \dfrac{dP}{dt} = -25$ cm/sec

$A = 2(1 + \sqrt{2})\left(\dfrac{P}{8}\right)^2$

$A = \dfrac{1}{32}(1 + \sqrt{2})P^2$

$\dfrac{dA}{dt} = \dfrac{1}{16}(1 + \sqrt{2})P\,\dfrac{dP}{dt}$

When $P = 192$, $\dfrac{dA}{dt} = \dfrac{1}{16}(1 + \sqrt{2})(192)(-25) \approx -724.264.$

The rate of decrease is about $724.264\ \text{cm}^2/\text{min}$.

So, the answer is C.

44. $F(x) = \int_0^x f(t)\,dt$

$F'(x) = f(x)$

$F''(x) = f'(x)$

$$\int_0^2 \left[x^2 + f''(x)\right] dx = \left[\frac{x^3}{3} + f'(x)\right]_0^2$$

$$= \left(\frac{8}{3} + F''(2)\right) - \left(F''(0)\right) = \frac{8}{3} + \frac{1}{3} - (-4) = 7$$

So, the answer is C.

45. $y = x^2 - x, (2, -1)$

$d(x) = \sqrt{(x-2)^2 + (x^2 - x + 1)^2}$

$d(x)$ has a relative minimum at $x = 1$.

$d(1) = \sqrt{2} \approx 1.414$

So, the answer is C.

Section 2

1. $N(t) = 30\left(1.1 - e^{-0.05t}\right)$

(a) $\dfrac{N(30) - N(0)}{30 - 0} \approx 0.777$ parts/day

1 pt: answer with units

(b) $N'(10) \approx 0.910$ parts/day. The number of parts produced per day by the worker is increasing at a rate of 0.910 parts/day after the tenth day on the job.

2 pts:
- 1 pt: computes $N'(10)$
- 1 pt: interpretation

(c) $N(t) = \dfrac{1}{30}\int_0^{30} N(t)\,dt$

$N(t) \approx 17.4626 \Rightarrow t \approx 13.159$ days

2 pts:
- 1 pt: sets $N(t)$ equal to the definite integral for average value
- 1 pt: answer

(d) $N(30) \approx 26.3061$

$N'(30) \approx 0.3347$

$L(t) = N(30) + N'(30)(t - 30)$

$33 = 26.3061 + 0.3347(t - 30) \Rightarrow t \approx 50.000$

The worker produces 33 parts on day 50.

4 pts:
- 2 pts: expression for $L(t)$
- 1 pt: sets $L(t)$ equal to 33
- 1 pt: answer

Reminder: Round answers to at least three decimal places to receive credit.

2. $v(t) = \left[8\sin\left(0.5t^2\right)\right] / (t - 8)$

(a) In $0 < t \le 4$, $v(t) = 0$ when $t \approx 2.5066$ and $t \approx 3.5449$.

$v(t)$ changes from negative to positive at time $t \approx 2.5066$.

$v(t)$ changes from positive to negative at time $t \approx 3.5449$.

Therefore, the particle changes direction at time $t \approx 2.507$ and $t \approx 3.545$.

3 pts: { 1 pt: considers $v(t) = 0$; 2 pts: answers with justification [where $v(t)$ changes sign] }

(b) $v(1) \approx -0.548 < 0$

$a(1) = v'(1) \approx -1.081 < 0$

The speed is increasing at time $t = 1$ because velocity and acceleration have the same sign at $t = 1$. So, the particle is speeding up.

1 pt: conclusion with justification [considers the signs of $v(1)$ and $a(1)$]

(c) $\int_0^3 v(t)\,dt \approx -1.154$ is the displacement of the particle over the time interval $0 \le t \le 3$.

$\int_0^3 |v(t)|\,dt = -\int_0^{2.5066} v(t)\,dt + \int_{2.5066}^3 v(t)\,dt \approx 2.017$ is the total distance traveled by the particle over the time interval $0 \le t \le 3$.

4 pts: { 2 pts: computes $\int_0^3 v(t)\,dt$ and interprets as displacement; 2 pts: computes $\int_0^3 |v(t)|\,dt$ and interprets as total distance traveled }

(d) $s(t) = t^2 - 2t$

$s'(t) = 2t - 2$

$v(t) = 2t - 2 \Rightarrow t \approx 0.817$

The two particles are moving with the same velocity at time $t \approx 0.817$.

1 pt: answer with justification [sets $v(t) = 2t - 2$]

Reminder: Round answers to at least three decimal places to receive credit.

3. (a) f' is positive on $(0, 3)$ and $(3, 5)$.

f' is decreasing on $(1, 3)$ and $(4, 6)$.

The graph of f is both increasing and concave downward on $(1, 3)$ and $(4, 5)$ because f' is positive and decreasing on these intervals.

2 pts: 1 pt: intervals; 1 pt: reason (considers where f' is positive and decreasing)

(b) f has a relative maximum at $x = 5$. This is the only critical point at which f' changes sign from positive to negative.

2 pts: 1 pt: answer; 1 pt: justification (considers where f' changes sign from positive to negative)

(c) The graph of f has points of inflection at $x = 1$ and $x = 4$ because f' changes from increasing to decreasing at these points. The graph of f has a point of inflection at $x = 3$ because f' changes from decreasing to increasing at this point.

2 pts: 1 pt: identifies $x = 1$, $x = 3$, and $x = 4$; 1 pt: reasons

(d) $\int_3^x f'(t)\,dt = f(x) - f(3)$

$f(x) = f(3) + \int_3^x f'(t)\,dt$

$f(x) = 6 + \int_3^x f'(t)\,dt$

$f(0) = 6 + \int_3^0 f'(t)\,dt = 6 - 9 = -3$

$f(5) = 6 + \int_3^5 f'(t)\,dt = 6 + 4 = 10$

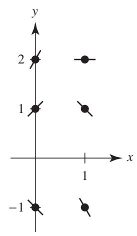
3 pts: 2 pts: expression for $f(x)$ using $f(3)$ and a definite integral; 1 pt: computes $f(0)$ and $f(5)$

4. $\dfrac{dy}{dx} = y - 2x$

(a) At $(0, 2)$, $\dfrac{dy}{dx} = 2$. At $(0, 1)$, $\dfrac{dy}{dx} = 1$.

At $(0, -1)$, $\dfrac{dy}{dx} = -1$. At $(1, 2)$, $\dfrac{dy}{dx} = 0$.

At $(1, 1)$, $\dfrac{dy}{dx} = -1$. At $(1, -1)$, $\dfrac{dy}{dx} = -3$.



2 pts: {1 pt: slopes where $x = 0$; 1 pt: slopes where $x = 1$}

(b) $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} - 2 = y - 2x - 2$

In Quadrant IV, $x > 0$ and $y < 0$,

so $\dfrac{d^2y}{dx^2} = y - 2x - 2 < 0$ for all (x, y) in Quadrant IV. Thus, all solution curves are concave downward in Quadrant IV.

2 pts: {1 pt: computes $\dfrac{d^2y}{dx^2}$; 1 pt: answer with reason (reasons with the signs of x and y in Quadrant IV)}

(c) $y(-2) = 4$

At $(-2, 4)$, $\dfrac{dy}{dx} = 4 - 2(-2) = 8 \neq 0$.

So, f has neither a relative minimum nor a relative maximum at $x = -2$.

2 pts: {1 pt: considers $\dfrac{dy}{dx}$ at $(-2, 4)$; 1 pt: conclusion with reason}

(d) $y = mx + b$ is a solution $\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}[mx + b]$

$$y - 2x = m$$
$$mx + b - 2x = m$$
$$(m - 2)x + (b - m) = 0$$
$$m - 2 = 0 \Rightarrow m = 2$$
$$b - m = 0 \Rightarrow b = m = 2$$

So, $y = mx + b$ is a solution when $m = 2$ and $b = 2$.

3 pts: {1 pt: uses $\dfrac{d}{dx}[mx + b] = m$; 1 pt: sets $y - 2x = m$; 1 pt: answer}

5. (a) $R'(5) \approx \dfrac{R(6) - R(4)}{6 - 4}$

$= \dfrac{202 - 306}{2}$

$= \dfrac{-104}{2}$

$= -52 \text{ gal/h}^2$

2 pts: { 1 pt: estimate; 1 pt: units }

(b) $\int_0^8 R(t)\,dt \approx 2\left[R(0) + R(2) + R(4) + R(6)\right]$

$= 2[610 + 442 + 306 + 202]$

$= 3120$ gallons

This is an overestimate because $R(t)$ is strictly decreasing on $0 \le t \le 8$.

3 pts: { 1 pt: sets up left Riemann sum; 1 pt: estimate; 1 pt: conclusion with reason }

(c) $10{,}000 - 3120 + \int_0^8 750e^{-t/3}\,dt = 6880 + \left[-2250e^{-t/3}\right]_0^8$

$= 6880 - 2250e^{-8/3} + 2250$

$= 9130 - \dfrac{2250}{e^{8/3}}$ gallons

2 pts: { 1 pt: definite integral; 1 pt: answer }

(d) $F(t) = R(t) \Rightarrow F(t) - R(t) = 0$

$F(0) - R(0) = 750 - 610 > 0$

$F(8) - R(8) = \dfrac{750}{e^{8/3}} - 130 < 0$

Because $F(t)$ and $R(t)$ are continuous on $0 \le t \le 8$, $F(t) - R(t)$ is continuous on $0 \le t \le 8$. So, by the Intermediate Value Theorem, there exists at least one time t in $0 \le t \le 8$ for which $F(t) - R(t) = 0$.

Thus, there exists at least one time t for which $F(t) = R(t)$.

2 pts: { 1 pt: considers $F(t) - R(t)$; 1 pt: answer with explanation (reasons with the Intermediate Value Theorem) }

6. (a) $V = \pi r^2 h = 4\pi h$

$$\frac{dV}{dt} = 4\pi\frac{dh}{dt} = 4\pi\left(\frac{1}{2}\sqrt{h}\right) = 2\pi\sqrt{h}$$

At $h = 9$, $\frac{dV}{dt} = 2\pi\sqrt{9} = 6\pi$ cubic feet per minute.

2 pts:
- 1 pt: $\frac{dV}{dt} = 4\pi\frac{dh}{dt}$
- 1 pt: answer with units

(b) $\frac{dh}{dt} = \frac{1}{2}\sqrt{h}$

$$\frac{d^2h}{dt^2} = \frac{1}{4\sqrt{h}}\frac{dh}{dt} = \frac{1}{4\sqrt{h}}\left(\frac{1}{2}\sqrt{h}\right) = \frac{1}{8}$$

Because $\frac{d^2h}{dt^2} = \frac{1}{8} > 0$ for all h and t, the rate of change of the height of the water is increasing when the height of the water is 4 feet.

3 pts:
- 1 pt: finds $\frac{d}{dh}\left[\frac{1}{2}\sqrt{h}\right]$
- 1 pt: $\frac{d^2h}{dt^2} = \frac{1}{4\sqrt{h}}\cdot\frac{dh}{dt}$
- 1 pt: answer with reason (uses the sign of $\frac{d^2h}{dt^2}$)

(c) $\frac{dh}{dt} = \frac{1}{2}\sqrt{h},\ h(0) = 2$

$$\frac{dh}{\sqrt{h}} = \frac{1}{2}\,dt$$

$$\int\frac{dh}{\sqrt{h}} = \int\frac{1}{2}\,dt$$

$$2\sqrt{h} = \frac{1}{2}t + C$$

$$2\sqrt{2} = C$$

$$2\sqrt{h} = \frac{1}{2}t + 2\sqrt{2}$$

$$\sqrt{h} = \frac{1}{4}t + \sqrt{2}$$

$$h = \left(\frac{1}{4}t + \sqrt{2}\right)^2$$

4 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration using initial condition
- 1 pt: $h(t)$

AP Practice Exam: Calculus BC

Section 1

1. $y = e^{-2x} + \cos 3x^2 - 4$

$\dfrac{dy}{dx} = -2e^{-2x} - 6x \sin 3x^2$

When $x = 0, \dfrac{dy}{dx} = -2.$

So, the answer is A.

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{(n+2)^2} - \frac{1}{n^2} \right)$

$$S_n = \left(\frac{1}{3^2} - 1\right) + \left(\frac{1}{4^2} - \frac{1}{2^2}\right) + \left(\frac{1}{5^2} - \frac{1}{3^2}\right) + \left(\frac{1}{6^2} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{(n+2)^2} - \frac{1}{n^2}\right)$$

$$= -1 - \frac{1}{4} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2}$$

$\displaystyle\lim_{n\to\infty} S_n = -\frac{5}{4}$

So, the answer is A.

3. $x(t) = \dfrac{4}{t^2 - 2}, y(t) = \dfrac{-2t}{t+1}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{(t+1)(-2) + 2t}{(t+1)^2}}{\frac{-8t}{(t^2-2)^2}} = \frac{-2(t^2-2)^2}{(t+1)^2(-8t)} = \frac{(t^2-2)^2}{4t(t+1)^2}$$

When $t = 1, \dfrac{dy}{dx} = \dfrac{1}{16}.$

$x(1) = -4, y(1) = -1$

An equation of the tangent line at $t = 1$ is

$$y + 1 = \frac{1}{16}(x + 4)$$

$$y = \frac{1}{16}x - \frac{3}{4}.$$

So, the answer is C.

4. $f''(x) = x^3(x-2)^2\sqrt{x+1}$

$f''(x) = 0$ when $x = 0, x = 2,$ and $x = -1.$

Interval:	$-1 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value:	$-\frac{1}{2}$	1	10
Sign of f'':	$f'' < 0$	$f'' > 0$	$f'' > 0$

The graph of f has an inflection point when $x = 0$ because f'' changes sign at $x = 0$. So, the answer is A.

5. $h(x) = \int_0^x \sqrt{\cos 4t}\, dt$

$h'(x) = \sqrt{\cos 4x}$

$1 + [h'(x)]^2 = 1 + \cos 4x$

$$s = \int_{-\pi/8}^{\pi/8} \sqrt{1 + \cos 4x}\, dx$$
$$= \int_{-\pi/8}^{\pi/8} \sqrt{1 + 2\cos^2 2x - 1}\, dx$$
$$= \sqrt{2}\int_{-\pi/8}^{\pi/8} \cos 2x\, dx$$

So, the answer is B.

6. I. $\sum_{k=1}^{\infty} \left(\frac{\pi}{3}\right)^k$ is a geometric series with $|r| = \frac{\pi}{3} > 1$.

So, the geometric series diverges.

II. $\lim_{k\to\infty} \sqrt[k]{\left(\frac{\pi}{3k}\right)^k} = \lim_{k\to\infty} \frac{\pi}{3k} = 0 < 1.$

So, $\sum_{k=1}^{\infty} \left(\frac{\pi}{3k}\right)^k$ converges by the Root Test.

III. $\lim_{k\to\infty} \sqrt[k]{\left(\frac{\pi k}{3}\right)^k} = \lim_{k\to\infty} \frac{\pi k}{3} = \infty > 1.$

So, $\sum_{k=1}^{\infty} \left(\frac{\pi k}{3}\right)^k$ diverges by the Root Test.

So, the answer is B.

7. $$\int_{2.2}^{3.2} f(x)\, dx \approx \frac{1}{4}\big(f(2.325) + f(2.575) + f(2.825) + f(3.075)\big)$$
$$\approx 0.25\left(e^{-(2.325)^2} + e^{-(2.575)^2} + e^{-(2.825)^2} + e^{-(3.075)^2}\right)$$

So, the answer is C.

8. $$\int \frac{5x^3 - 4x^2 - 3x - 2}{x - 1} dx = \int \left(5x^2 + x - 2 - \frac{4}{x - 1}\right) dx \quad \text{(by long division)}$$
$$= \frac{5}{3}x^3 + \frac{1}{2}x^2 - 2x - 4\ln(x - 1) + C$$

So, the answer is A.

9. $$\int_0^{\infty} e^{px} dx = \lim_{b\to\infty} \int_0^b e^{px}\, dx$$
$$= \lim_{b\to\infty} \left[\frac{1}{p}e^{px}\right]_0^b$$
$$= \lim_{b\to\infty} \left[\frac{1}{p}e^{pb} - \frac{1}{p}\right]$$

When $p > 0$, $\lim_{b\to\infty} \left[\frac{1}{p}e^{pb} - \frac{1}{p}\right] = \infty.$

When $p < 0$, $\lim_{b\to\infty} \left[\frac{1}{p}e^{pb} - \frac{1}{p}\right] = -\frac{1}{p}.$

When $p = 0$, $\lim_{b\to\infty} \int_0^b dx = \lim_{b\to\infty} [x]_0^b = \lim_{b\to\infty} b = \infty.$

So, $\int_0^{\infty} e^{px}\, dx$ diverges when $p \geq 0$. So, the answer is D.

10. $\mathbf{v}(t) = \left\langle \ln(t + 3), \frac{7t}{2t^2 + 6} \right\rangle$

$$\int \ln(t + 3)\, dt = t\ln(t + 3) + 3\ln(t + 3) - t - 3 + C_1 \quad \text{(integration by parts)}$$
$$= (t + 3)\ln(t + 3) - t + C_2$$

$$\int \frac{7t}{2t^2 + 6}\, dt = \frac{7}{4}\ln(2t^2 + 6) + C_3 = \frac{7}{4}\ln(t^2 + 3) + \frac{7}{4}\ln(2) + C_3$$

A position vector has the form $\left\langle (t + 3)\ln(t + 3) - t + C_2, \frac{7}{4}\ln(t^2 + 3) + C_4 \right\rangle$.

So, the answer is C.

11. $\frac{1}{4}\int_0^4 (-2x^2 + 8x)\, dx = \frac{1}{4}\left[-\frac{2}{3}x^3 + 4x^2\right]_0^4$

$$= \frac{1}{4}\left[-\frac{128}{3} + 64\right]$$
$$= \frac{16}{3}$$

So, the answer is C.

12. $f(x) = \ln(e^x + x)$

$$f'(x) = \frac{e^x + 1}{e^x + x}$$

$$f''(x) = \frac{(e^x + x)(e^x) - (e^x + 1)^2}{(e^x + x)^2}$$

$$f''(0) = \frac{1 - 4}{1} = -3$$

$$\frac{f''(0)}{2!} = -\frac{3}{2}$$

So, the answer is B.

13. $f(x) = \int_0^x t^2 e^t\, dt$

$f'(x) = x^2 e^x$ by the Fundamental Theorem of Calculus.

So, the answer is D.

14. $g(x) = \int_0^x \sin(-t^2)\, dt$

$$g'(x) = \sin(-x^2)$$

$$g''(x) = -2x\cos(-x^2)$$

On the interval $0 < x < \frac{\pi}{3}$, $\sin(-x^2) < 0$ and $\cos(-x^2) > 0$. So, $g'(x) < 0$ and $g''(x) < 0$ on this interval. Thus, g is decreasing, and the graph of g is concave downward.

So, the answer is D.

15. $\sum_{k=1}^{\infty} \frac{(-1)^{k-3}}{2^{k+5}}$ is a geometric series with $r = -\frac{1}{2}$.

$$S = \frac{\frac{1}{2^6}}{1 - \left(-\frac{1}{2}\right)} = \frac{\frac{1}{64}}{\frac{3}{2}} = \frac{1}{96}$$

So, the answer is D.

16. $\lim_{x\to 1} \frac{xe^{2x} - (e^x)^2}{2x^2 - x - 1}$ yields an indeterminate form, $\frac{0}{0}$, so L'Hôpital's Rule applies.

$$\lim_{x\to 1} \frac{xe^{2x} - e^{2x}}{2x^2 - x - 1} = \lim_{x\to 1} \frac{2xe^{2x} + e^{2x} - 2e^{2x}}{4x - 1} = \frac{e^2}{3}$$

So, the answer is A.

17. $A = \int_{-\pi/5}^{\pi/5} \frac{1}{2}\left[(3 + 2\cos 5\theta)^2 - 1^2\right] d\theta$

$$= \int_0^{\pi/5} (9 + 12\cos 5\theta + 4\cos^2 5\theta - 1)\, d\theta$$
$$= \int_0^{\pi/5} (8 + 12\cos 5\theta + 2(1 + \cos 10\theta))\, d\theta$$
$$= \int_0^{\pi/5} (10 + 12\cos 5\theta + 2\cos 10\theta)\, d\theta$$
$$= \left[10\theta + \frac{12}{5}\sin 5\theta + \frac{1}{5}\sin 10\theta\right]_0^{\pi/5}$$
$$= 2\pi$$

So, the answer is B.

18. $V = \pi \int_{-1}^{1/2} \left[\left(e^{-x^2+x+2}\right)^2\left(\sqrt{1-2x}\right)^2 - \left(-\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}\right)^2\right] dx$

$= \pi \int_{-1}^{1/2} \left(\left[e^{-2x^2+2x+4}(1-2x)\right] - \left[\frac{4}{3}x^2 - \frac{4}{3}x + \frac{1}{3}\right]\right) dx$

$= \pi \left[\frac{1}{2}e^{-2x^2+2x+4} - \frac{4}{9}x^3 + \frac{2}{3}x^2 - \frac{1}{3}x\right]_{-1}^{1/2}$

$= \pi\left[\left(\frac{1}{2}e^{9/2} - \frac{1}{18} + \frac{1}{6} - \frac{1}{6}\right) - \left(\frac{1}{2} + \frac{4}{9} + \frac{2}{3} + \frac{1}{3}\right)\right]$

$= \pi\left(\frac{1}{2}e^{9/2} - 2\right)$

$= \frac{\pi}{2}\left(e^{9/2} - 4\right)$

So, the answer is A.

19. Because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$,

$\sin x^2 = x^2 - \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^5}{5!} - \frac{\left(x^2\right)^7}{7!} + \cdots$

$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

So, the answer is A.

20. $g(x) = \int_0^{x^2} te^{-(t+1)^2}\,dt$

$g'(x) = x^2e^{-\left(x^2+1\right)^2}(2x) = 2x^3e^{-\left(x^2+1\right)^2}$

$g''(x) = 2x^3e^{-\left(x^2+1\right)^2}(-2)\left(x^2+1\right)(2x) + 6x^2e^{-\left(x^2+1\right)^2}$

$= 2x^2e^{-\left(x^2+1\right)^2}\left[-4x^2\left(x^2+1\right) + 3\right]$

$= -2x^2e^{-\left(x^2+1\right)^2}\left(4x^4 + 4x^2 - 3\right)$

$= -2x^2e^{-\left(x^2+1\right)^2}\left(2x^2 + 3\right)\left(2x^2 - 1\right)$

$g'' = 0$ when $x = 0$, $x = -\frac{\sqrt{2}}{2}$, and $x = \frac{\sqrt{2}}{2}$.

Interval:	$-\infty < x < -\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2} < x < 0$	$0 < x < \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} < x < \infty$
Sign of g'':	$f'' < 0$	$f'' > 0$	$f'' > 0$	$f'' < 0$

g has points of inflection at $x = -\frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{2}$.

So, the answer is B.

21. $\frac{dP}{dt} = 2P\left(1 - \frac{P}{4}\right), P(0) = 3, h = \frac{1}{2}$

$P(0.5) \approx 3 + 6\left(1 - \frac{3}{4}\right)\left(\frac{1}{2}\right) = 3 + \frac{3}{4} = \frac{15}{4}$

$P(1) \approx \frac{15}{4} + \frac{15}{2}\left(1 - \frac{15}{16}\right)\left(\frac{1}{2}\right) = \frac{15}{4} + \frac{15}{64} \approx 3.75 + 0.25 \approx 4$ mice

So, the answer is B.

22. $P_3(x) = -4 + x + \frac{3}{2}x^2 - \frac{3}{6}x^3$

$= -4 + x + \frac{3}{2}x^2 - \frac{1}{2}x^3$

So, the answer is D.

23. $V = \pi r^2 h = k$. Surface area of a cylinder is minimized when its diameter equals its height: $2r = h$.

So, $V = \pi r^2(2r) = k \Rightarrow 2\pi r^3 = k \Rightarrow r = \sqrt[3]{\frac{k}{2\pi}}$.

The area of the bottom of the cylinder is

$A = \pi r^2 = \pi\sqrt[3]{\frac{k^2}{4\pi^2}} = \sqrt[3]{\frac{k^2\pi^3}{4\pi^2}} = \sqrt[3]{\frac{\pi k^2}{4}}$.

So, the answer is B.

24. $\frac{dy}{dt} = kyt^3$

$\frac{dy}{y} = kt^3 dt$

$\ln|y| = \frac{kt^4}{4} + C_1$

$y = e^{(kt^4/4)+C_1}$

$y = e^{kt^4/4} \cdot e^{C_1}$

$y = C_2 e^{kt^4/4}$

So, the answer is B.

25. $u = e^x + 1, du = e^x\,dx, e^x = u - 1$

$\int \frac{e^{2x}}{(e^x + 1)^2}\,dx = \int \frac{u - 1}{u^2}\,du$

$= \int \left(\frac{1}{u} - \frac{1}{u^2}\right) du$

$= \ln|u| + \frac{1}{u} + C$

$= \ln(e^x + 1) + \frac{1}{e^x + 1} + C$

So, the answer is C.

26. $\lim_{k\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{k\to\infty}\left|\frac{(x - 2)^{2(k+1)}}{2^{k+6}(k + 1)^2} \cdot \frac{2^{k+5}k^2}{(x - 2)^{2k}}\right|$

$= \lim_{k\to\infty}\left|\frac{(x - 2)^2 k^2}{2(k + 1)^2}\right| = \frac{(x - 2)^2}{2}$

The power series converges when

$\frac{(x - 2)^2}{2} < 1 \Rightarrow (x - 2)^2 < 2 \Rightarrow -\sqrt{2} < x - 2 < \sqrt{2} \Rightarrow 2 - \sqrt{2} < x < 2 + \sqrt{2}$.

The radius of convergence is $R = \sqrt{2}$.

So, the answer is C.

27. $\int_0^1 3xe^{2x}dx$

$u = x \Rightarrow du = dx$

$dv = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

$$3\int_0^1 xe^{2x}dx = 3\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1$$
$$= 3\left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right)$$
$$= \frac{3}{4}\left(e^2 + 1\right)$$

So, the answer is B.

28. I. $$\sum_{k=1}^{\infty}\left(\frac{1}{5k+3} - \frac{1}{5k+1}\right) = \sum_{k=1}^{\infty} -\frac{2}{(5k+3)(5k+1)}$$
$$= -2\sum_{k=1}^{\infty}\frac{1}{25k^2+20k+3}$$

$$\lim_{k\to\infty}\frac{\frac{1}{25k^2+20k+3}}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{k^2}{25k^2+20k+3} = \frac{1}{25}$$

Therefore, $-2\sum_{k=1}^{\infty}\frac{1}{25k^2+20k+3}$ converges by a limit comparison with the convergent p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

II. $$\sum_{k=1}^{\infty}\left(\frac{1}{5k+3} - \frac{1}{k+1}\right) = \sum_{k=1}^{\infty}\frac{-4k-2}{(5k+3)(k+1)}$$
$$= -2\sum_{k=1}^{\infty}\frac{2k+1}{5k^2+8k+3}$$

$$\lim_{k\to\infty}\frac{\frac{2k+1}{5k^2+8k+3}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{2k^2+k}{5k^2+8k+3} = \frac{2}{5}$$

Therefore, $-2\sum_{k=1}^{\infty}\frac{2k+1}{5k^2+8k+3}$ diverges by a limit comparison with the divergent p-series $\sum_{k=1}^{\infty}\frac{1}{n}$.

III. $\sum_{k=1}^{\infty}\left(\frac{1}{k+3} - \frac{1}{k+1}\right)$ is a convergent telescoping series. (A comparison test can also be used to show convergence.)
So, the answer is D.

29. $x(t) = \frac{t+1}{\ln(t+2)},\ y(t) = 2t\sin\frac{t^2}{3}$

$x'(2) \approx 0.33109,\ y'(2) \approx 3.19848$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \approx \frac{3.19848}{0.33109} \approx 9.6604$$

So, the answer is D.

30. $V = \frac{1}{3}\pi r^2 h,\ r = \frac{2}{3}h \Rightarrow h = \frac{3}{2}r$

$$V = \frac{1}{3}\pi r^2\left(\frac{3}{2}r\right) = \frac{1}{2}\pi r^3$$

$$\frac{dV}{dt} = \frac{3}{2}\pi r^2\frac{dr}{dt}$$

$$\frac{\frac{dV}{dt}}{\frac{3}{2}\pi r^2} = \frac{dr}{dt}$$

When $h = 2,\ r = \frac{2}{3}(2) = \frac{4}{3}$.

So, $\frac{dr}{dt} = \frac{0.01}{\frac{3}{2}\pi\left(\frac{4}{3}\right)^2} \approx 0.00119$ m/min.

So, the answer is C.

31. From the graph, $f(0) = 1$ and $f'(0) = 0$. Because f is concave downward at $x = 0$, $f''(0) < 0$.

So, $f''(0) < f'(0) < f(0)$.

So, the answer is D.

32. $\arccos y = (x-1)^2 \Rightarrow y = \cos(x-1)^2,\ y = (\ln x)^2$

Intersection points occur at $x \approx 0.38166$ and $x \approx 2.02468$.

$$A = \int_{0.38166}^{2.02468}\left[\cos(x-1)^2 - (\ln x)^2\right]dx$$
$$\approx 1.179$$

So, the answer is C.

33. $$P_2(x) = g(1) + \frac{g'(1)}{1}(x-1) + \frac{g''(1)}{2}(x-1)^2$$
$$= 4 - 2(x-1) + \frac{3}{2}(x-1)^2$$

$$\int_{0.2}^{1.3} e^x\left[4 - 2(x-1) + \frac{3}{2}(x-1)^2\right]dx \approx 10.964$$

So, the answer is C.

34. $V = \pi \int_{\pi/6}^{9\pi/4} \left(e^{\sin x} + 1\right)^2 dx \approx 121.354$

So, the answer is C.

35. $f(36) = 377,\ f'(x) \geq -14$

The least possible value of $f(42)$ is $377 + (-14)(6) = 293$.

(Note: An equation of the line through $(36, 377)$ with slope $m = -14$ is $y = -14x + 881$.)

So, the answer is C.

36. $r = 1 + \cos 2\theta,\ r = 2\cos\theta - \sqrt{4\cos^2\theta - 3}$

Intersection points occur at $\theta \approx \pm 0.5093$.

$$A = \frac{1}{2}\int_{-0.5093}^{0.5093} \left[(1 + \cos 2\theta)^2 - \left(2\cos\theta - \sqrt{4\cos^2\theta - 3}\right)^2\right] d\theta$$
$$= 1.078$$

So, the answer is C.

37. $g(t) = \dfrac{25x + 25}{x^2 + 2x + 2}$

$\int_0^{14} g(t)\,dt + 3 \approx 59 + 3 = 62$ cubic meters

So, the answer is B.

38. $s(t) = -5t + \int_0^t (2u)^{u+1}\,du$

$$a = \frac{s(4) - s(0)}{4 - 0} = \frac{1}{4}\left(-20 + \int_0^4 (2u)^{u+1}\,du\right) \approx 2502.039$$

$v(t) = -5 + (2t)^{t+1}$

$b = v(2) = -5 + (4)^3 = 59$

$\dfrac{a}{b} \approx 42.407$

So, the answer is D.

39. Because $g'(7) = 0$ and $g''(7) = 4 > 0$, g has a local minimum at $x = 7$.

So, the answer is D.

40. $\dfrac{dy}{dx} = \dfrac{2x^2y}{3},\ y(0) = 5$

$$\frac{dy}{y} = \frac{2}{3}x^2\,dx$$
$$\ln|y| = \frac{2}{9}x^3 + C_1$$
$$y = e^{(2/9)x^3 + C_1}$$
$$y = Ce^{(2/9)x^3}$$
$$y = 5e^{(2/9)x^3} \quad (C = 5)$$
$$y(0.3) = 5e^{(2/9)(0.3)^3} \approx 5.030$$

So, the answer is B.

41. $a_n = -\dfrac{1}{2n + 1}$

Because $\lim_{n\to\infty} a_n = 0$, the nth-term test for divergence is inconclusive. Consider $-\sum_{n=1}^{\infty} \dfrac{1}{2n + 1}$ and compare to the divergent p-series, $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n + 1} = \lim_{n\to\infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2}$$

Therefore, $-\sum_{n=1}^{\infty} \dfrac{1}{2n + 1}$ diverges by the Limit Comparison Test. So, the answer is A.

42. $\dfrac{dy}{dx} = \sqrt{1 - y^2},\ \left(0, -\dfrac{1}{7}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{4\sqrt{3}}{7}\right)$

$$\frac{dy}{\sqrt{1 - y^2}} = dx$$
$$\arcsin y = x + C$$
$$\arcsin\left(-\frac{1}{7}\right) = C$$
$$-0.1433 \approx C$$
$$y = \sin(x - 0.1433)$$

So, the answer is A.

43. $v(t) = \left(\frac{5}{4}\right)^t \sin\left(\frac{t}{2}\right)$

$a(t) = \left(\frac{5}{4}\right)^t \cdot \frac{1}{2}\cos\left(\frac{t}{2}\right) + \sin\left(\frac{t}{2}\right)\left(\frac{5}{4}\right)^t \cdot \ln\left(\frac{5}{4}\right)$

Using the graph of $a(t)$, $a(t)$ is increasing on $(0, 1.679)$ and $(7.962, 14{,}245)$. So, the answer is A.

44. $v(t) = t^3 - 8t^2 + 6t - [5t/(t+1)]$

$v(t) > 0$ on $(0, 0.2645)$ and $(7.2566, 8)$

$v(t) < 0$ on $(0.2645, 7.2566)$

$$\text{Total distance} = \int_0^8 |v(t)|\,dt = \int_0^{0.2645} v(t)\,dt - \int_{0.2645}^{7.2566} v(t)\,dt + \int_{7.2566}^{8} v(t)\,dt$$
$$\approx 208.71$$

So, the answer is D.

45. $u = x^2 \qquad dv = f''(x)\,dx$

$du = 2x\,dx \qquad v = f'(x)$

$$\int x^2 f''(x)\,dx = x^2 f'(x) - 2\int x f'(x)\,dx$$

$u = x \qquad dv = f'(x)\,dx$

$du = dx \qquad v = f(x)$

$$\int x^2 f''(x)\,dx = x^2 f'(x) - 2\left(x f(x) - \int f(x)\,dx\right)$$

$$\text{So, } \frac{1}{2}\int_1^6 x^2 f''(x)\,dx = \frac{1}{2}\left[x^2 f'(x) - 2x f(x)\right]_1^6 + \int_1^6 f(x)\,dx$$
$$= \frac{1}{2}\left[(36f'(6) - 12f(6)) - (f'(1) - 2f(1))\right] + 40$$
$$= \frac{1}{2}\left[(36)(6) - (12)(18) + (2)(3)\right] + 40$$
$$= 43$$

So, the answer is B.

Section 2

1. $S(t) = 1500e^{-t^3/110}$

(a) $S'(2) \approx -152.158 \text{ tons/h}^2$

The rate at which sand is poured into the tank is decreasing by 152.158 tons per hour per hour at time $t = 2$ hours.

2 pts: {1 pt: answer; 1 pt: interpretation}

(b) $\int_0^8 S(t)\,dt \approx 6410.673$ tons

2 pts: {1 pt: sets up definite integral; 1 pt: answer with units}

(c) $S(t) = \frac{1}{8}\int_0^8 S(t)\,dt$

$S(t) = 801.334 \Rightarrow t \approx 4.101$ hours

2 pts: {1 pt: sets $S(t)$ equal to the definite integral for average value; 1 pt: answer}

(d) $S(5) \approx 481.476 < 500$

At time $t = 5$, the rate at which sand is poured into the tank is less than the rate at which sand is removed from the tank. So, the amount of sand in the tank at time $t = 5$ is decreasing.

3 pts: {1 pt: computes $S(5)$; 1 pt: considers the removal rate (500 tons per hour); 1 pt: conclusion with explanation}

Reminder: Round answers to at least three decimal places to receive credit.

2. $\frac{dx}{dt} = t^2 + \cos 4t^2,\ x(0) = 4,\ y(0) = 0$

(a) $x(3) = x(0) + \int_0^3 x'(t)\,dt \approx 4 + 9.272 = 13.272$

$y(3) = 0.5$

The position of the particle at $t = 3$ is $(13.272, 0.5)$.

3 pts: { 1 pt: sets up definite integral in finding $x(3)$; 1 pt: uses initial condition in finding $x(3)$; 1 pt: answer }

(b) For $1 < t < 2$, $\frac{dy}{dt} = \frac{2+1}{2-1} = 3$.

Slope $= \frac{y'(t)}{x'(t)} = \frac{3}{x'(t)} = 3 \Rightarrow x'(t) = 1$

For $1 < t < 2$, $x'(t) = 1$ when $t \approx 1.069$.

2 pts: { 1 pt: computes $\frac{dy}{dt}$ using graph; 1 pt: answer [finds where $x'(t) = 1$] }

(c) Speed $= \sqrt{\left[\left[x'(1.5)\right]^2 + \left[y'(1.5)\right]^2\right]}$

$\approx \sqrt{(1.3389)^2 + 3^2}$

≈ 3.285

1 pt: answer with justification

(d) Distance $= \int_1^4 \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt$

$= \int_1^2 \sqrt{\left[x'(t)\right]^2 + 3^2}\,dt + \int_2^4 \sqrt{\left[x'(t)\right]^2 + \left(-\frac{3}{2}\right)^2}\,dt$

$\approx 3.9195 + 18.9912$

≈ 22.911

3 pts: { 1 pt: expression for distance; 1 pt: splits into two definite integrals; 1 pt: answer }

Reminder: Round answers to at least three decimal places to receive credit.

3. (a) $G'(9) \approx \frac{G(10) - G(8)}{10 - 8} = \frac{20.9 - 16.5}{2} = \frac{4.4}{2} = 2.2$ MB/sec

2 pts: { 1 pt: estimate; 1 pt: units }

(b) G is differentiable and therefore continuous on $[4, 8]$.

$\frac{G(8) - G(4)}{8 - 4} = \frac{16.5 - 12.5}{4} = 1$

By the Mean Value Theorem, there exists at least one time t in $[4, 8]$ for which $G'(t) = 1$.

2 pts: { 1 pt: computes the average rate of change on $[4, 8]$; 1 pt: conclusion with explanation (reasons with the Mean Value Theorem) }

(c) $\frac{1}{12}\int_0^{12} G(t)\,dt \approx \frac{1}{12} \cdot 4\left[G(2) + G(6) + G(10)\right]$

$= \frac{1}{3}(6.3 + 14.8 + 20.9)$

$= \frac{1}{3}(42)$

$= 14$ MB

$\frac{1}{12}\int_0^{12} G(t)\,dt$ is the average number of megabytes downloaded over the time interval $[0, 12]$ seconds.

3 pts: { 1 pt: sets up a midpoint sum; 1 pt: approximation; 1 pt: interpretation }

(d) $V(t) = 32 - 32e^{-0.11t}$

$V'(t) = 3.52e^{-0.11t}$

$V'(10) = 3.52e^{-1.1} = \frac{3.52}{e^{1.1}}$ MB/sec

2 pts: { 1 pt: computes $V'(t)$; 1 pt: answer with units }

4. $\frac{dy}{dx} = 3x - \frac{y}{2}$

(a) $\frac{d^2y}{dx^2} = 3 - \frac{1}{2}\frac{dy}{dx} = 3 - \frac{3}{2}x + \frac{y}{4}$

In Quadrant II, $x < 0$ and $y > 0$.

So, $\frac{d^2y}{dx^2} = 3 - \frac{3}{2}x + \frac{y}{4} > 0$ for all (x, y) in Quadrant II.

Thus, all solution curves are concave upward in Quadrant II.

2 pts:
- 1 pt: computes $\frac{d^2y}{dx^2}$
- 1 pt: answer with reason (uses the signs of x and y in Quadrant II)

(b) $y = mx + b$ is a solution.

$$\frac{dy}{dx} = \frac{d}{dx}[mx + b]$$

$$3x - \frac{y}{2} = m$$

$$3x - \frac{1}{2}(mx + b) = m$$

$$\left(3 - \frac{1}{2}m\right)x - \left(\frac{1}{2}b + m\right) = 0$$

$$3 - \frac{1}{2}m = 0 \Rightarrow m = 6$$

$$\frac{1}{2}b + m = 0 \Rightarrow b = -2m = -12$$

So, y is a solution when $m = 6$ and $b = -12$.

3 pts:
- 1 pt: uses $\frac{d}{dx}[mx + b] = m$
- 1 pt: sets $3x - \frac{y}{2} = m$
- 1 pt: answer

(c) $y = f(x)$ is the particular solution given $f(0) = 3$.

(i) At $(0, 3)$, $\frac{dy}{dx} = 3(0) - \frac{3}{2} = -\frac{3}{2} \neq 0$.

So, f has neither a relative minimum nor relative maximum at $x = 0$.

(ii) $h = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) \approx 3 + \left(0 - \frac{3}{2}\right)\left(\frac{1}{2}\right) = 3 - \frac{3}{4} = \frac{9}{4}$$

$$f(1) \approx \frac{9}{4} + \left(\frac{3}{2} - \frac{9}{8}\right)\left(\frac{1}{2}\right) = \frac{9}{4} + \left(\frac{3}{8}\right)\left(\frac{1}{2}\right) = \frac{9}{4} + \frac{3}{16} = \frac{39}{16}$$

4 pts:
- 1 pt: considers $\frac{dy}{dx}$ at $(0, 3)$
- 1 pt: conclusion with reason
- 1 pt: approximation of $f(0.5)$ using Euler's Method
- 1 pt: approximation of $f(1)$ using Euler's Method

5. $r = 5, r = 2 + 6\sin\theta$, intersection at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

(a) Area $= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(2 + 6\sin\theta)^2 - 5^2\right]d\theta$

3 pts: { 1 pt: constant and limits; 2 pts: integrand }

(b) $x = r\cos\theta = (2 + 6\sin\theta)\cos\theta$

$\frac{dx}{d\theta} = -(2 + 6\sin\theta)\sin\theta + 6\cos^2\theta$

At $\theta = \pi$, $\frac{dx}{d\theta} = 6$.

$y = r\sin\theta = (2 + 6\sin\theta)\sin\theta$

$\frac{dy}{d\theta} = (2 + 6\sin\theta)\cos\theta + 6\sin\theta\cos\theta$

At $\theta = \pi$, $\frac{dy}{d\theta} = -2$.

At $\theta = \pi$, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2}{6} = -\frac{1}{3}$.

3 pts: { 1 pt: finds $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$; 1 pt: represents $\frac{dy}{dx}$ as $\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$; 1 pt: answer }

(c) $r = 2 + 6\sin\theta$, $\frac{dr}{dt} = 2$

$\frac{dr}{dt} = 6\cos\theta\frac{d\theta}{dt}$

When $\theta = \frac{\pi}{3}$,

$2 = 6\cos\left(\frac{\pi}{3}\right)\frac{d\theta}{dt}$

$2 = 3\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{2}{3}$ rad/sec.

3 pts: { 2 pts: computes $\frac{dr}{dt}$; 1 pt: answer with units }

6. $f(0) = 0$

$f^{(n)}(0) = -\left(\frac{(-1)^{n+2} - 1}{2}\right)^{n(n+1)/2}$

(a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$

3 pts: { 2 pts: first four terms; 1 pt: general term }

(b) $\lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)-1}}{(2(n+1)-1)!}}{\frac{x^{2n-1}}{(2n-1)!}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+1)(2n)} = 0 < 1$ for all x

Therefore, the interval of convergence is $(-\infty, \infty)$.

4 pts: { 1 pt: sets up ratio; 1 pt: computes limit of ratio; 1 pt: observes this limit is < 1 for all x; 1 pt: interval of convergence }

(c) $\left|P_5\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{\left(\frac{1}{4}\right)^{11}}{11!} = \frac{1}{(4^{11})(11!)}$

2 pts: { 1 pt: uses the next term as error bound; 1 pt: answer }