

AP[®] Exam Practice Questions for Chapter 8

1. Evaluate each series.

$$\text{I: } \sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

Because $p = 1/3$ and $0 < p < 1$, the series diverges.

$$\text{II: } \sum_{n=1}^{\infty} \left(\frac{e}{\sin 2} \right)^n$$

Because $|r| = e/\sin 2 \approx 2.99 > 1$, the series diverges.

$$\text{III: } \sum_{n=1}^{\infty} \frac{5}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{5}{(n+2)!} \cdot \frac{(n+1)!}{5} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+2} \right| = 0 < 1$$

By the Ratio Test, the series converges.
So, the answer is C.

$$2. \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n}$$

$$a_{n+1} = \frac{(n+1)^3}{4^{n+1}} < \frac{n^3}{4^n}$$

$$\lim_{n \rightarrow \infty} \frac{n^3}{4^n} = 0$$

By the Alternating Series Test, the series converges.

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n n^3}{4^n} \right|} = \lim_{n \rightarrow \infty} \frac{(n^3)^{1/n}}{(4^n)^{1/n}} = \lim_{n \rightarrow \infty} \frac{n^{3/n}}{4} = \frac{1}{4} < 1$$

By the Root Test, the series converges absolutely.
So, the answer is B.

$$5. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$= 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!} - \dots$$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \dots$$

$$x^2 \cos x^2 = x^2 - \frac{x^2 \cdot x^4}{2!} + \frac{x^2 \cdot x^8}{4!} - \frac{x^2 \cdot x^{12}}{6!} + \frac{x^2 \cdot x^{16}}{8!} - \dots$$

$$= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \frac{x^{18}}{8!} - \dots$$

So, the answer is C.

$$3. 1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \dots + \frac{(\ln 3)^n}{n!} + \dots$$

Let $x = \ln 3$.

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$= e^x$$

$$= e^{\ln 3}$$

$$= 3$$

So, the answer is C.

$$4. \sum_{n=2}^{\infty} \frac{5(n+3)}{(n-1)^p}$$

Use the Limit Comparison test to compare with

$$\sum_{n=2}^{\infty} \frac{1}{n^{p-1}}, \text{ which converges for } p-1 > 1 \Rightarrow p > 2.$$

$$\lim_{n \rightarrow \infty} \left| \frac{5(n+3)}{(n-1)^p} \cdot \frac{n^{p-1}}{1} \right| = \lim_{n \rightarrow \infty} \left| \frac{5n^p + 15n^{p-1}}{(n-1)^p} \right| = 5$$

Because $\lim_{n \rightarrow \infty}$ is finite and positive, $\sum_{n=2}^{\infty} \frac{5(n+3)}{(n-1)^p}$

converges when $p > 2$.

So, the answer is C.

$$6. \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n^3 + 1} \right| \Rightarrow a_n = \frac{1}{3n^3 + 1}$$

$$a_{n+1} < 0.001$$

$$\frac{1}{3(n+1)^3 + 1} < \frac{1}{1000}$$

$$1000 < 3(n+1)^3 + 1$$

$$999 < 3(n+1)^3$$

$$333 < (n+1)^3$$

$$n > -1 + \sqrt[3]{333}$$

$$n > 5.9313$$

$$\Rightarrow n = 6$$

So, the answer is B.

$$8. (a) P_1(x) = g(3) + g'(3)(x-3) = 50 + \frac{160}{3}(x-3)$$

$$\text{So, } g(3.1) \approx 50 + \frac{160}{3}(3.1-3) \approx 55.\bar{3}.$$

Because $g'(x)$ is increasing on $[2, 4]$, $g''(3) > 0$. So, $g(x)$ is concave upward at $x = 3$, and the tangent line at $x = 3$ will lie below the graph of g . So, the approximation is less than the actual value of $g(3.1)$.

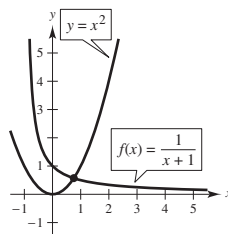
$$\begin{aligned} (b) P_3(x) &= g(3) + g'(3)(x-3) + \frac{g''(3)}{2!}(x-3)^2 + \frac{g'''(3)}{3!}(x-3)^3 \\ &= 50 + \frac{160}{3}(x-3) + \frac{141}{4 \cdot 2!}(x-3)^2 + \frac{21}{3!}(x-3)^3 \\ &= 50 + \frac{160}{3}(x-3) + \frac{141}{8}(x-3)^2 + \frac{7}{2}(x-3)^3 \end{aligned}$$

$$\begin{aligned} \text{So, } g(3.1) &\approx 50 + \frac{160}{3}(3.1-3) + \frac{141}{8}(3.1-3)^2 + \frac{7}{2}(3.1-3)^3 \\ &\approx 55.51308. \end{aligned}$$

$$\begin{aligned} (c) |g(3.1) - P_3(3.1)| &= |R_n(3.1)| = \left| \frac{g^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right| \\ &= \left| \frac{g^{(4)}(z)}{4!}(x-3)^4 \right| = \frac{1123}{8 \cdot 4!}(3.1-3)^4 = 0.0005849 < 0.0006 \end{aligned}$$

$$7. \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

$$\text{Graph } f(x) = \frac{1}{1+x} \text{ and } y = x^2.$$



The graphs intersect at $x = 0.755$.

So, the answer is C.

4 pts: $\begin{cases} 2 \text{ pts: first-degree Taylor polynomial at } x = 3 \\ 1 \text{ pt: approximation of } g(3.1) \\ 1 \text{ pt: answer ("less than") with reason} \end{cases}$

3 pts: $\begin{cases} 2 \text{ pts: third-degree Taylor polynomial at } x = 3 \\ 1 \text{ pt: approximation of } g(3.1) \end{cases}$

2 pts: $\begin{cases} 1 \text{ pt: uses the fourth term as an error bound} \\ 1 \text{ pt: computes error bound to compare} \end{cases}$

Reminders: Round each answer to at least three decimal places to receive credit for the approximation.

You do not need to simplify the coefficients in these Taylor polynomials.

In these approximations, be sure to write " $g(3.1) \approx$ " rather than " $g(3.1) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

9. (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \cdots + \frac{(-x)^n}{n!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x}{n!} + \cdots$$

$$xe^{-x} = x - x \cdot x + \frac{x \cdot x^2}{2!} - \frac{x \cdot x^3}{3!} + \cdots + \frac{(-1)^n \cdot x \cdot x^n}{n!} + \cdots$$

$$= x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \cdots + \frac{(-1)^n x^{n+1}}{n!} + \cdots$$

So, the first four nonzero terms are $x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}$.

(b) $\lim_{x \rightarrow 0} \frac{f(x) - x + x^2}{x^3} = \lim_{x \rightarrow 0} \frac{xe^{-x} - x + x^2}{x^3} = \frac{1}{2}$

(c) $g(x) = \int_0^x te^{-t} dt$

$$= \int_0^x \left(t - t^2 + \frac{t^3}{2!} - \frac{t^4}{3!} + \cdots + \frac{(-1)^n t^{n+1}}{n!} + \cdots \right) dt$$

$$= \int_0^x \left(t - t^2 + \frac{t^3}{2} - \frac{t^4}{6} + \cdots + \frac{(-1)^n t^{n+1}}{n!} + \cdots \right) dt$$

$$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{8}t^4 - \frac{1}{30}t^5 + \cdots + \frac{(-1)^n}{(n+2)n!}t^{n+2} + \cdots \right]_0^x$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{30}x^5 + \cdots + \frac{(-1)^n}{(n+2)n!}x^{n+2} + \cdots$$

So, $g\left(\frac{1}{5}\right) \approx \frac{1}{2}\left(\frac{1}{5}\right)^2 - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{8}\left(\frac{1}{5}\right)^4$.

(d) $|a_{n+1}| < \frac{1}{90,000}$

$$\left| \frac{(-1)^{n+1} x^{n+3}}{(n+3)(n+1)!} \right| < \frac{1}{90,000}$$

$$\frac{(1/5)^6}{(6)(4!)} < \frac{1}{90,000}$$

$$\frac{1}{2,250,000} < \frac{1}{90,000}$$

2 pts: $\begin{cases} 1 \text{ pt: first four terms of Taylor series} \\ 1 \text{ pt: general term} \end{cases}$

1 pt: answer with justification [using results from part (a)]

4 pts: $\begin{cases} 2 \text{ pts: first four nonzero terms} \\ 1 \text{ pt: general term} \\ 1 \text{ pt: approximation of } g(1/5) \end{cases}$

Reminders: This approximation does not need to be simplified.

Write " $g(1/5) \approx$ " rather than " $g(1/5) =$."

Because this is an approximation, a point may be deducted if an equal sign is used.

2 pts: $\begin{cases} 1 \text{ pt: correct form of error bound} \\ 1 \text{ pt: computes error bound to compare} \end{cases}$

Reminder: This error bound does not need to be simplified.

10. (a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$

$$\begin{aligned} f(x) = \cos x^2 &= 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots + \frac{(-1)^n (x^2)^{2n}}{(2n)!} + \dots \\ &= 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \dots + \frac{(-1)^n x^{4n}}{(2n)!} + \dots \end{aligned}$$

(b) $\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{4(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{(-1)^n x^{4n}} \right| = \lim_{n \rightarrow \infty} x^4 \left| \frac{1}{(2n+2)(2n+1)} \right| = 0$

Because the series converges for all x , $R = \infty$.

(c) $\cos x^2 \approx 1 - \frac{x^4}{2} + \frac{x^8}{24}$

$$\cos(1) \approx 1 - \frac{(1)^4}{2} + \frac{(1)^8}{24} = \frac{13}{24}$$

$$\begin{aligned} |a_{n+1}| &= \left| \frac{x^{12}}{6!} \right| \\ &= \left| \frac{(1)^{12}}{6!} \right| \\ &= \frac{1}{720} < \frac{1}{500} \end{aligned}$$

3 pts: $\begin{cases} 2 \text{ pts: first four terms of series} \\ 1 \text{ pt: general term} \end{cases}$

Reminder: You do not need to simplify the coefficients in these Taylor polynomials.

3 pts: $\begin{cases} 1 \text{ pt: sets up ratio} \\ 1 \text{ pt: computes limit of ratio} \\ 1 \text{ pt: finds radius of convergence} \end{cases}$

3 pts: $\begin{cases} 1 \text{ pt: approximation of } f(1) \text{ using first three terms of series} \\ 1 \text{ pt: uses the fourth term as an error bound} \\ 1 \text{ pt: computes error bound to compare} \end{cases}$

Reminders: Write " $\cos(1) \approx$ " rather than " $\cos(1) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

This error bound does not need to be simplified.

11. (a) $f(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots$

$$f'(x) = -2x + \frac{4x^3}{2} - \frac{6x^5}{6} + \frac{8x^7}{24} - \frac{10x^9}{120} + \dots$$

$$= -2x + 2x^3 - x^5 + \frac{1}{3}x^7 - \frac{1}{12}x^9 + \dots$$

$$f''(x) = -2 + 6x^2 - 5x^4 + \frac{7}{3}x^6 - \frac{3}{4}x^8 + \dots$$

So, $f'(0) = 0$ and $f''(0) = -2$.

Because $f'(0) = 0$, f has a critical value at $x = 0$.

Because $f''(0) < 0$, f is concave downward at $x = 0$.

So, by the Second Derivative Test, f has a relative maximum at $x = 0$.

(b) $f(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$

$$f(1) = 1 - (1) + \frac{(1)^4}{2!} - \frac{(1)^6}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

Use the Alternating Series Test to find the error.

$$|a_{n+1}| = \left| \frac{x^8}{4!} \right| = \frac{(1)^8}{4!} = \frac{1}{24} < \frac{1}{10}$$

So, $f(1) = \frac{1}{2!} - \frac{1}{3!}$ with an error $= \frac{1}{24} < \frac{1}{10}$.

(c) $y' + 2xy = 0$

$$y' = -2xy$$

$$\frac{dy}{dx} = -2xy$$

$$\frac{1}{y} dy = -2x dx$$

$$\int \frac{1}{y} dy = \int -2x dx$$

$$\ln |y| = -x^2 + C$$

$$y = e^{-x^2 + C}$$

$$y = Ce^{-x^2} = e^{-x^2} \text{ when } C = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots = f(x)$$

So, $f(x) = e^{-x^2}$ is a solution of the differential equation $y' + 2xy = 0$.

4 pts: $\begin{cases} 1 \text{ pt: finds } f'(0) \\ 1 \text{ pt: finds } f''(0) \\ 2 \text{ pts: conclusion with reasoning} \end{cases}$

Reminder: Explicitly identify each function by name. Referring to “it,” “the function,” or “the graph” will not receive credit on the exam.

3 pts: $\begin{cases} 1 \text{ pt: approximation of } f(1) \text{ using first four terms of series} \\ 1 \text{ pt: uses the fifth term as an error bound} \\ 1 \text{ pt: computes error bound to compare} \end{cases}$

2 pts: $\begin{cases} 1 \text{ pt: computes } y' = f'(x) \text{ using given power series} \\ 1 \text{ pt: shows that this differential equation is satisfied} \end{cases}$

12. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{n(x-1)}{n+1} \\ &= x-1\end{aligned}$$

$$|x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

When $x = 0$: $\frac{(-1)^{n+1}(-1)^n}{n} = \frac{(-1)^{2n+1}}{n} = -\frac{1}{n}$

The series diverges (divergent p -series).

When $x = 2$: $\frac{(-1)^{n+1}(1)^n}{n} = \frac{(-1)^{n+1}}{n}$

The series converges by the Alternating Series Test.

So, the interval of convergence is $(0, 2]$.

(b) $f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

$$\begin{aligned}g(x) = f'(x) &= (1-0) - (x-1) + (x-1)^2 - (x-1)^3 + \dots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots + (-1)^n(x-1)^n\end{aligned}$$

(c) $g(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots + (-1)^n(x-1)^n$

$$= \frac{1}{x}$$

(d) $h(x) = f(x^3 + 1)$

$$\begin{aligned}h'(x) &= f'(x^3 + 1) \cdot (3x^2) \\ &= g(x^3 + 1)(3x^2) \\ &= \frac{1}{x^3 + 1} \cdot 3x^2 \\ &= \frac{3x^2}{x^3 + 1}\end{aligned}$$

4 pts: $\left\{ \begin{array}{l} 1 \text{ pt: sets up ratio and computes} \\ \text{limit of ratio} \\ 1 \text{ pt: finds interior of interval of} \\ \text{convergence} \\ 2 \text{ pts: considers both endpoints,} \\ \text{shows analysis for interval} \\ \text{of convergence} \end{array} \right.$

2 pts: $\left\{ \begin{array}{l} 1 \text{ pt: first three terms of series} \\ 1 \text{ pt: general terms} \end{array} \right.$

1 pt: answer

2 pts: finds $h'(x) = f'(x^3 + 1) = g(x^3 + 1)$
[using g from part (c)]

13. (a) Because $g'(2) = 4$, the equation of the tangent line at $(2, 3)$ is

$$y - 3 = 4(x - 2)$$

$$y = 4x - 5.$$

$$\text{So, } g(2.4) \approx 4(2.4) - 5 = 4.6.$$

Because $g''(2) > 0$, $g(x)$ is concave upward at $x = 2$, and the tangent line at $x = 2$ will lie below the graph of g . So, the approximation is less than $g(2.4)$.

$$\begin{aligned} \text{(b) } \int_2^{2.4} g'(x) \, dx &= 0.2[g'(2.1) + g'(2.3)] \\ &= 0.2(5 + 8) \\ &= 2.6 \end{aligned}$$

$$\begin{aligned} g(2.4) &\approx g(2) + \int_2^{2.4} g'(x) \, dx \\ &= 3 + 2.6 \\ &= 5.6 \end{aligned}$$

$$\begin{aligned} \text{(c) } g(2.2) &\approx g(2) + 0.2[g'(2)] \\ &= 3 + 0.2(4) = 3.8 \\ g(2.4) &\approx g(2.2) + 0.2[g'(2.2)] \\ &= 3.8 + 0.2(7) = 5.2 \end{aligned}$$

$$\begin{aligned} \text{(d) } P_2(x) &= g(2) + g'(2)(x - 2) + \frac{g''(2)(x - 2)^2}{2!} \\ &= 3 + 4(x - 2) + \frac{20}{2}(x - 2)^2 \\ &= 3 + 4(x - 2) + 10(x - 2)^2 \\ \text{So, } g(2.4) &\approx 3 + 4(2.4 - 2) + 10(2.4 - 2)^2 \\ &= 6.2. \\ \text{Error: } \left| \frac{g'''(2)}{3!}(2.4 - 2)^3 \right| &= \frac{6\left(\frac{2}{5}\right)^3}{125} = \frac{8}{125} < \frac{1}{10} \end{aligned}$$

$$\begin{aligned} 2 \text{ pts: } &\begin{cases} 1 \text{ pt: tangent line equation and approximation of } \\ g(2.4) \\ 1 \text{ pt: answer with reason} \end{cases} \end{aligned}$$

Reminders: This approximation does not need to be simplified.

Write " $g(2.4) \approx$ " rather than " $g(2.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

$$\begin{aligned} 2 \text{ pts: } &\begin{cases} 1 \text{ pt: midpoint Riemann sum approximation} \\ 1 \text{ pt: approximation of } g(2.4) \end{cases} \end{aligned}$$

Reminder: Write " $g(2.4) \approx$ " rather than

" $g(2.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

$$\begin{aligned} 2 \text{ pts: } &\begin{cases} 1 \text{ pt: Euler's Method with two steps} \\ 1 \text{ pt: answer} \end{cases} \end{aligned}$$

Reminders: This approximation does not need to be simplified.

Write " $g(2.4) \approx$ " rather than " $g(2.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

$$\begin{aligned} 3 \text{ pts: } &\begin{cases} 1 \text{ pt: second-degree Taylor polynomial} \\ 1 \text{ pt: approximation of } g(2.4) \\ 1 \text{ pt: finds error bound using the third term and compares} \end{cases} \end{aligned}$$

Reminders: This approximation does not need to be simplified.

Write " $g(2.4) \approx$ " rather than " $g(2.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used.