AP® Exam Practice Questions for Chapter 5

1. To find which graph is a slope field for $\frac{dy}{dx} = y - \frac{x}{5}$, evaluate the derivative at selected points.

At $(0, 1)$, $\frac{dy}{dx} = 1$.

At $(1, 0)$, $\frac{dy}{dx} = -\frac{1}{5}$.

At $(5, 0)$, $\frac{dy}{dx} = -1$,

So, the answer is B.

2. $\frac{dP}{dt} = kP$

$\int \frac{1}{P} = \int k\ dt$

$\ln|P| = kt + C_1$

$P = Ce^{kt}$

Use $(15, 3)$ and $C = 1$ to find k.

$3 = 1e^{15k}$

$\ln 3 = 15k$

$\frac{\ln 3}{15} = k$

So, the answer is B.

3. $y' = ky$

$\frac{1}{y}y' = k$

$\int \frac{1}{y}y' = \int k$

$\ln|y| = kx + C_1$

$y = Ce^{kx}$

Use $f(0) = 8$ and $f(6) = 2$ to find C and k.

$8 = Ce^{k(0)} \Rightarrow C = 8$

$2 = Ce^{k(6)}$

$2 = 8e^{6k}$

$\frac{1}{4} = e^{6k}$

$\frac{1}{6} \ln \frac{1}{4} = k$

Because $f(x) = Ce^{kx} = 8e^{(x/6)\ln(1/4)}$, the answer is A.

4. $\frac{dy}{dx} = 2xy^2$

$\frac{1}{y^2}\ dy = 2x\ dx$

$\int \frac{1}{y^2}\ dy = \int 2x\ dx$

$-\frac{1}{y} = x^2 + C$

Use $y(-1) = 2$ to find C.

$-\frac{1}{2} = (-1)^2 + C \Rightarrow C = -\frac{3}{2}$

$-\frac{1}{y} = x^2 - \frac{3}{2} \Rightarrow y = -\frac{2}{2x^2 + 3}$

$y(2) = -\frac{2}{2(2)^2 - 3} = -\frac{2}{5}$

So, the answer is C.

5. $\frac{dy}{dx} = \frac{3y}{x}$

$\frac{1}{y}\ dy = \frac{3}{x}\ dx$

$\int \frac{1}{y}\ dy = 3\int \frac{1}{x}\ dx$

$\ln|y| = 3\ln|x| + C_1$

$\ln|y| = \ln|x^3| + C_1$

$y = Cx^3$

Use $y(1) = -1$ to find C.

$-1 = C(1)^3$

$-1 = C$

The solution of the differential equation is $y = -x^3$.

So, the answer is B.

6. To find which graph is a slope field, evaluate the derivative at selected points.

When $y = 3$, $\frac{dy}{dx}$ should be 0.

When $y = 2$, $\frac{dy}{dx}$ should be > 0.

When $y = 4$, $\frac{dy}{dx}$ should be < 0.

So, the answer is C.

7. Use Euler's Method with $y(0) = -3$, and $h = \frac{1}{3}$ to find $y(1)$.

$y\left(\frac{1}{3}\right) \approx -3 + \frac{1}{3}(-2) = -\frac{11}{3}$

$y\left(\frac{2}{3}\right) \approx -\frac{11}{3} + \frac{1}{3}\left(-\frac{28}{9}\right) = -\frac{127}{27}$

$y(1) \approx -\frac{127}{27} + \frac{1}{3}(-4.753) \approx -6.288$

So, the answer is A.

8. (a) $\frac{dy}{dt} = 0.5y$

$\frac{1}{y}\,dy = 0.5\,dt$

$\int \frac{1}{y}\,dy = \int 0.5\,dt$

$\ln|y| = 0.5t + C_1$

$y = Ce^{0.5t}$

Use $(0, 200)$ to find C.

$200 = Ce^{0.5(0)} \Rightarrow C = 200$

So, $y = 200e^{0.5t}$.

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

(b) $\frac{1}{10}\int_0^{10} 200e^{0.5t}\,dt = \frac{1}{10}\cdot 2\int_0^{10} 200e^{0.5t}(0.5)\,dt$

$= \frac{1}{5}\left[200e^{0.5t}\right]_0^{10}$

$= 40\left(e^5 - 1\right)$

≈ 5896.526 bacteria

2 pts:
- 1 pt: definite integral
- 1 pt: answer with units (no work)

Reminders: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator

Be sure to round the answer to at least three decimal places to receive credit on the exam.

Note: Importing an incorrect particular solution from part (a) can still earn the first point in part (b), but not the answer point.

(c) $\frac{1}{20}\int_0^{10} 200e^{0.5t}\,dt$ bacteria/hour

2 pts:
- 1 pt: expression
- 1 pt: units

9. (a) $\dfrac{dy}{dx} = \dfrac{1}{xy}$

$$\frac{d^2y}{dx^2} = \frac{xy(0) - \left[x\dfrac{dy}{dx} + y\right]}{(xy)^2}$$

$$= -\frac{x\left(\dfrac{1}{xy}\right) + y}{x^2y^2}$$

$$= -\frac{\dfrac{1}{y} + y}{x^2y^2}$$

$$= -\frac{1 + y^2}{x^2y^3}$$

At the point $(1, 2)$, $\dfrac{d^2y}{dx^2} = -\dfrac{1 + (2)^2}{(1)^2(2)^3} = -\dfrac{5}{8}$.

2 pts: implicit differentiation to find d^2y/dx^2 (in terms of x and y); evaluates

(b) Find $\dfrac{dy}{dx}$ at the point $(1, 2)$.

$$\frac{dy}{dx} = \frac{1}{xy} = \frac{1}{(1)(2)} = \frac{1}{2}$$

So, the equation of the tangent line is

$$y - 2 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{3}{2}.$$

$$f(1.1) \approx \frac{1}{2}(1.1) + \frac{3}{2} = 2.05$$

Because $f''(1.1) < 0$, the approximation for $f(1.1) \approx 2.05$ is greater than $f(1.1)$.

2 pts:
- 1 pt: finds equation of tangent line using dy/dx at $(1, 2)$ and approximates $f(1.1)$
- 1 pt: answer ("greater than") with reason (appeals to the concavity of f)

(c) $\dfrac{dy}{dx} = \dfrac{1}{xy}$

$$y\,dy = \frac{1}{x}\,dx$$

$$\int y\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}y^2 = \ln|x| + C$$

Use $f(1) = 2$ to find C.

$$\frac{1}{2}(2) = \ln(1) + C$$

$$2 = 0 + C$$

$$2 = C$$

So, the solution is

$$\frac{1}{2}y^2 = \ln|x| + 2$$

$$y^2 = 2\ln|x| + 4$$

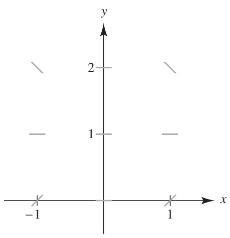
$$y = \sqrt{2\ln|x| + 4}.$$

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

10. (a)



2 pts: 1 pt: correct slopes at point where $x = 0$ or $y = 1$ (horizontal line segments)
1 pt: correct slopes at other points

(b) $\frac{dy}{dx} = x^2(1 - y) > 0$

$$1 - y > 0$$
$$-y > -1$$
$$y < 1$$

So, the slopes are positive when $y < 1$.

1 pt: answer with justification

(c) $$\frac{dy}{dx} = x^2(1 - y)$$

$$\frac{1}{y - 1}\,dy = -x^2\,dx$$

$$\int \frac{1}{y - 1}\,dy = -\int x^2\,dx$$

$$\ln|y - 1| = -\frac{1}{3}x^3 + C_1$$

$$y - 1 = e^{(-1/3)x^3}e^{C_1}$$

$$y = 1 + Ce^{(-1/3)x^3}$$

Use $f(0) = 2$ to find C.

$$2 = 1 + Ce^{(-1/3)(0)^3}$$
$$2 = 1 + C$$
$$C = 1$$

So, the solution is $f(x) = 1 + e^{(-1/3)x^3}$.

6 pts: 1 pt: separation of variables
2 pts: antiderivatives
1 pt: constant of integration
1 pt: uses initial condition
1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

11. (a) Use Euler's Method with $y' = \frac{2x}{y}, f(1) = 2,$ and $h = 0.2$ to approximate $y(1.4)$.

$$y_1 = y_0 + hF(x_0, y_0) = 2 + 0.2\left[\frac{2(1)}{2}\right] = 2.2$$

$$y_2 = y_1 + hF(x_1, y_1) = 2.2 + 0.2\left[\frac{2(1.2)}{2.2}\right]$$

So, $y(1.4) = 2.2 + 0.2\left[\frac{2.4}{2.2}\right]$.

3 pts: $\begin{cases} \text{2 pts: Euler's Method with two steps} \\ \text{1 pt: answer } [\text{approximation of } f(1.4)] \end{cases}$

Reminders: The answer does *not* need to be simplified. Leaving the answer as $2.2 + 0.2\left[\frac{2(1.2)}{2.2}\right]$ is recommended.

Be sure to write "$f(1.4) \approx$" rather than "$f(1.4) =$." Because this is an approximation, a point may be deducted if an equal sign is used. In general, equating two quantities that are not truly equal will result in a one point deduction on a free-response question.

(b) $\frac{dy}{dx} = \frac{2x}{y}$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 = 2x^2 + C$$

Use $(1, 2)$ to find C.

$$(2)^2 = 2(1)^2 + C$$

$$4 = 2 + C$$

$$C = 2$$

Because $y^2 = 2x^2 + 2$, the solution is $y = \sqrt{2x^2 + 2}$, where the domain is $(-\infty, \infty)$.

6 pts: $\begin{cases} \text{1 pt: separation of variables} \\ \text{1 pt: antiderivatives} \\ \text{1 pt: constant of integration} \\ \text{1 pt: uses initial condition} \\ \text{1 pt: particular solution (solves for } y) \\ \text{1 pt: states domain of } y \end{cases}$

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

12. (a) Use Euler's Method, $\frac{dy}{dx} = xy$, $f(1) = 1$, and $h = 0$ to find $f(1.2)$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1[(1)(1)] = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1[(1.1)(1.1)]$

So, $f(1.2) = 1.1 + 0.1[(1.1)(1.1)]$.

2 pts: {1 pt: Euler's Method with two steps; 1 pt: answer [approximation of $f(1.2)$]}

Reminders: The answer does *not* need to be simplified. Leaving the answer as $1.1 + 0.1[(1.1)(1.1)]$ is recommended.

Be sure to write "$f(1.2) \approx$" rather than "$f(1.2) =$." Because this is an approximation, a point may be deducted if an equal sign is used. In general, equating two quantities that are not truly equal will result in a one point deduction on a free-response question.

(b) $\frac{dy}{dx} = xy$

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y(1) = x(xy) + y = x^2y + y$$

On the interval $[1, 1.2]$, x is positive. Because $\frac{dy}{dx} = xy$, $\frac{dy}{dx}$ and y have the same sign on $[1, 1.2]$. Because $y = f(1) = 1$, $\frac{dy}{dx}$ and y are both positive on $[1, 1.2]$. Therefore, $\frac{d^2y}{dx^2} > 0$, so the solution y is concave upward on this interval and any tangent line approximation will be *below* the curve y. So, the approximation found in part (a) is less than $f(1.2)$.

2 pts: {1 pt: implicit differentiation to find d^2y/dx^2 (in terms of x and y); 1 pt: answer with reason (appeals to the concavity of y on this interval)}

(c) $\frac{dy}{dx} = xy$

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C_1$$

$$y = e^{(1/2)x^2}e^{C_1}$$

$$y = Ce^{(1/2)x^2}$$

Use $(1, 1)$ to find C.

$$1 = Ce^{(1/2)(1)^2}$$

$$1 = Ce^{1/2} \Rightarrow C = e^{-1/2}$$

So, the solution is

$$y = e^{-1/2}e^{(1/2)x^2} = e^{(1/2)x^2 - 1/2} = e^{0.5x^2 - 0.5}.$$

5 pts: {1 pt: separation of variables; 1 pt: antiderivatives; 1 pt: constant of integration; 1 pt: uses initial condition; 1 pt: particular solution (solves for y)}

Notes: 2 points max if no constant of integration present

0 points if no separation of variables

13. At $y = 100$, $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$

$$= \frac{1}{10}(100)\left(1 - \frac{100}{1000}\right)$$

$$= 10\left(\frac{9}{10}\right)$$

$$= 9.$$

At $y = 200$, $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$

$$= \frac{1}{10}(200)\left(1 - \frac{200}{1000}\right)$$

$$= 20\left(\frac{8}{10}\right)$$

$$= 16.$$

Because $\frac{dy}{dt} = 9$ when $y = 100$ is less than $\frac{dy}{dt} = 16$ when the disease is spreading faster when 200 people have the disease.

3 pts:
- 2 pts: Computes dy/dt at $y = 100$ and $y = 200$
- 1 pt: conclusion with reason

Reminder: To avoid the risk of an arithmetic mistake, these answers do *not* need to be simplified.

(b) $\frac{dy}{dt} = \frac{1}{10}y\left(1 - \frac{y}{1000}\right)$, where $L = 1000$ and $k = \frac{1}{10}$.

So, $y = \frac{1000}{1 + Ce^{(-1/10)t}}$.

Use $(0, 100)$ to find C.

$$100 = \frac{1000}{1 + Ce^{(-1/10)(0)}}$$

$$1 + C = 10$$

$$C = 9$$

So, a model for the population is $y = \frac{1000}{1 + 9e^{(-1/10)t}}$.

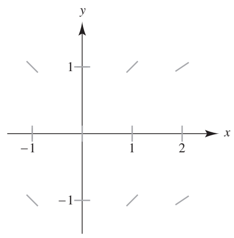
5 pts:
- 2 pts: general solution of the logistic differential equation
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution

Note: It is very unlikely that you would be asked to actually solve a logistic differential equation on a free-response question. (This equation is separable, but finding its solution involves a partial fraction decomposition.) However, you may be asked to approximate a solution to a logistic differential equation using Euler's method, a Taylor polynomial, or a slope field.

(c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1000}{1 + 9e^{(-1/10)t}} = \frac{1000}{1 + 0} = 1000$

1 pt: answer

14. (a)



2 pts: slopes of line segments

(b) $\dfrac{dy}{dx} = \dfrac{x}{y^2} = xy^{-2}$

$$\frac{d^2y}{dx^2} = x\left[-2y^{-3}\frac{dy}{dx}\right] + y^{-2}(1)$$

$$= -\frac{2x}{y^3}\left(\frac{x}{y^2}\right) + \frac{1}{y^2}$$

$$= \frac{-2x^2 + y^3}{y^5}$$

2 pts: implicit differentiation to find d^2y/dx^2 (in terms of x and y)

(c) $\dfrac{dy}{dx} = \dfrac{x}{y^2}$

$$y^2\,dy = x\,dx$$

$$\int y^2\,dy = \int x\,dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

Use $y(0) = 2$ to find C.

$$\frac{1}{3}(2)^3 = \frac{1}{2}(0)^2 + C \Rightarrow C = \frac{8}{3}$$

So, the solution is

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + \frac{8}{3}$$

$$y^3 = \frac{3}{2}x^2 + 8 \Rightarrow y = \sqrt[3]{\frac{3}{2}x^2 + 8}.$$

5 pts:
- 1 pt: separation of variables
- 1 pt: antiderivatives
- 1 pt: constant of integration
- 1 pt: uses initial condition
- 1 pt: particular solution (solves for y)

Notes: 2 points max if no constant of integration present

0 points if no separation of variables