AP® Exam Practice Questions for Chapter 3

1. $f(x) = 4x^3 + 6x^2 - 7x - 9$

$$f'(x) = 12x^2 + 12x - 72 = 0$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x = -3, 2$$

The critical numbers of $f(x)$ are $x = -3$ and $x = 2$.

So, the answer is B.

2. Evaluate each statement.

I: Because the function is strictly monotonic and increasing, $f'(x) > 0$ on the entire number line.
The statement is true.

II: Because f is concave downward when $x < 1$, $f''(x) < 0$ on the interval $(-\infty, 1)$.
The statement is true.

III: Because f is concave upward when $x > 1$, $f''(x) > 0$ on the interval $(1, \infty)$.
The statement is true.

So, the answer is D.

3. $s(t) = -t^3 + 3t^2 + 9t + 5$

$$s'(t) = -3t^2 + 6t + 9$$
$$s''(t) = -6t + 6$$

$$s''(t) = 0$$
$$-6t + 6 = 0$$
$$6t = 6$$
$$t = 1$$

So, $t = 1$ is a point of inflection of $s(t)$.

Use $s'(t)$ to find the velocity at $t = 1$.

$$s'(1) = -3(1)^2 + 6(1) + 9$$
$$= 12$$

The maximum velocity is 12 feet per second.

So, the answer is B.

4. Evaluate each statement.

A: The point $(4, 1)$ appears to be a relative minimum, but there may be another number c on $[2, 6]$ for which $g'(c) = 0$.
The statement may not be true.

B: The point $(6, 7)$ appears to be a relative maximum, but there may be another number c on $[2, 6]$ for which $g'(c) = 0$.
The statement may not be true.

C: Because g is continuous and differentiable on $[2, 6]$ and $g(2) = g(6)$, then there is at least one number c in $(2, 6)$ such that $g'(c) = 0$.
By Rolle's Theorem, this statement must be true.

D: The graph of g appears to be decreasing on $(2, 4)$, but there may be a point on $(2, 4)$ at which $g'(x) = 0$.
The statement may not be true.

So, the answer is C.

5. $f(2) = 0$

$f'(2) < 0$ because f is decreasing at $x = 2$.

$f''(2) > 0$ because the graph of f is concave upward at $x = 2$.

Therefore, $f'(2) < f(2) < f''(2) \Rightarrow f''(2) > f(2)$.

So, the answer is D.

6. $y = \arctan 4x$

Let $u = 4x \Rightarrow du = 4\,dx$.

$$dy = \frac{du}{1 + u^2}$$
$$= \frac{4\,dx}{1 + (4x)^2}$$
$$= \frac{4}{1 + 16x^2}\,dx$$

So, the answer is A.

7. $f(x) = \ln(x - 3)$

$f'(x) = \dfrac{1}{x - 3}$

By the Mean Value Theorem, $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(8) - f(4)}{8 - 4} = \dfrac{\ln 5 - \ln 1}{4} = \dfrac{\ln 5}{4}$

$f'(x) = f'(c)$

$\dfrac{1}{x - 3} = \dfrac{\ln 5}{4}$

$x - 3 = \dfrac{4}{\ln 5}$

$x = \dfrac{4}{\ln 5} + 3$

$x \approx 5.485$

So, the answer is A.

8. (a) Because $f'(x) > 0$ when $x < 4$, f is increasing on the interval $(-\infty, 4)$.

2 pts: $\begin{cases} \text{1 pt: answer (ignore inclusion of endpoint)} \\ \text{1 pt: reason } [\text{appeals to where } f'(x) > 0] \end{cases}$

(b) Yes. Because f is continuous and $f'(x)$ changes from positive to negative at $x = 4$, f has a relative maximum at $x = 4$.

2 pts: answer with reason [appeals to $f'(x)$ changing from positive to negative at $x = 4$]

Reminder: In such an explanation, use $f'(x)$ in your reason (explain using a derivative). Merely reasoning with f (appealing to where f changes from increasing to decreasing) may not receive credit on the exam.

(c) Because f is continuous, $f(2) = 4$, $f''(x) < 0$ on $(-\infty, 4)$, and $f''(x) > 0$ on $(4, \infty)$, the point of inflection is at $x = 4$.

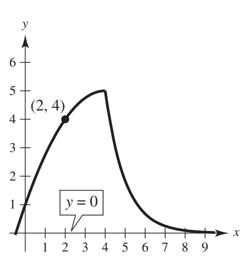
2 pts: answer with reason [appeals to $f''(x)$ changing sign at $x = 4$]

Reminder: Be sure to use $f''(x)$ in your reason (explain using a derivative) rather than reasoning with the concavity of f.

(d) No, $f(x)$ is not differentiable on $(3, 5)$.

1 pt: answer with reason

(e) Answers will vary.
Sample answer:



2 pts: $\begin{cases} \text{1 pt: graph reflects appropriate increasing/decreasing and concavity behavior} \\ \text{1 pt: appropriate behavior at } x = 4 \text{ [relative max, } f'(x) \text{ undefined]} \end{cases}$

Reminder: In the explanations throughout this question, be sure to explicitly identify each function by name. For example, referring to $f'(x)$ in part (a) as "it" or "the function" may not receive credit on the exam because there are three functions involved in the analysis of this question.

9. (a) $f(x) = \frac{x^3}{2} - \sin x + 1$

$f'(x) = \frac{3}{2}x^2 - \cos x$

Using a graphing utility,

$f'(x) = \frac{3}{2}x^2 - \cos x = 0$ when $x \approx \pm 0.7108.$

So, the relative extrema of f occur at $x \approx \pm 0.7108.$

3 pts: {1 pt: computes $f'(x)$; 2 pts: answers (finds critical numbers from calculator, no work needed)}

Note: You would be expected to compute/work with $f'(x)$ here in justifying the critical points. Merely obtaining the critical points from your calculator would not receive full credit on the exam.

Reminder: Round each answer to at least three decimal places to receive credit on the exam.

(b) $f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right)^3 - \sin\left(\frac{\pi}{2}\right) + 1$

$= \frac{\pi^3}{16} - 1 + 1$

$= \frac{\pi^3}{16}$

$f'\left(\frac{\pi}{2}\right) = \frac{3}{2}\left(\frac{\pi}{2}\right)^2 - \cos\frac{\pi}{2}$

$= \frac{3\pi^2}{8}$

Tangent line: $y - \frac{\pi^3}{16} = \frac{3\pi^2}{8}\left(x - \frac{\pi}{2}\right)$

$y = \frac{3\pi^2}{8}x - \frac{3\pi^3}{16} - \frac{\pi^3}{16}$

$y = \frac{3\pi^2}{8}x - \frac{\pi^3}{8}$

4 pts: {1 pt: finds $f\left(\frac{\pi}{2}\right)$; 2 pts: finds $f'\left(\frac{\pi}{2}\right)$; 1 pt: equation of tangent line, solves for y}

(c) Using the tangent line approximation,

$f(1.5) = \frac{3\pi^2}{8}(1.5) - \frac{\pi^3}{8} \approx 1.676.$

The actual value of

$f(1.5) = \frac{(1.5)^3}{2} - \sin(1.5) + 1 \approx 1.690.$

So, the tangent line approximation is an underestimate of $f(1.5)$.

2 pts: {1 pt: approximates $f(1.5)$; 1 pt: compares approximation with $f(1.5)$, conclusion}

Note: An alternate explanation may be to identify that the tangent line at $x = \pi/2$ is below the graph of f. To see this, analyze the sign of $f''(\pi/2)$ to determine the concavity of f at $x = \pi/2$.

Reminders: When using the tangent line to *approximate* $f(1.5)$, be sure to write "$f(1.5) \approx$" rather than "$f(1.5) =$." Because this is an approximation, a point may be deducted if an equal sign is used. In general, equating two quantities that are not truly equal will result in a one point deduction on a free-response question.

Be sure to round each answer to at least three decimal places to receive credit on the exam, and avoid premature rounding in intermediate steps.

Be sure your calculator is in *radian* mode.

10. (a) $f(x) = 2x + \cos 2x$

$$\begin{aligned} f'(x) &= 2 + (-\sin 2x)(2) \\ &= 2 - 2\sin 2x = 0 \\ 2\sin 2x &= 2 \\ \sin 2x &= 1 \\ x &= \frac{\pi}{4} \end{aligned}$$

Check $x = \frac{\pi}{4}$:

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= 2\left(\frac{\pi}{4}\right) + \cos\left(2 \cdot \frac{\pi}{4}\right) \\ &= \frac{\pi}{2} + \cos\frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

Check the endpoints of $[0, \pi]$:

$$\begin{aligned} f(0) &= 2(0) + \cos(2 \cdot 0) \\ &= \cos 0 \\ &= 1 \end{aligned}$$

$$f(\pi) = 2(\pi) + \cos(2\pi) = 2\pi + 1$$

Because $f(0) < f\left(\frac{\pi}{4}\right) < f(\pi)$, the maximum value of f is $2\pi + 1$.

5 pts:
- 1 pt: computes $f'(x)$ (applies Chain Rule)
- 1 pt: finds critical number [sets $f'(x) = 0$ and solves]
- 3 pts: find values of f at critical number and at endpoints of interval; answer

(b) Because f is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$, the conditions for the Mean Value Theorem are satisfied.

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{(2\pi + 1) - 1}{\pi} = 2$$

$$\begin{aligned} f'(x) &= f'(c) \\ 2 - 2\sin 2x &= 2 \\ -2\sin 2x &= 0 \\ \sin 2x &= 0 \\ x &= \frac{\pi}{2} \end{aligned}$$

By the Mean Value Theorem, $x = \frac{\pi}{2}$.

4 pts:
- 1 pt: identifies that both necessary conditions are satisfied on this interval
- 1 pt: finds the average rate of change on $[0, \pi]$ (difference quotient)
- 1 pt: sets $f'(x)$ equal to this difference quotient
- 1 pt: answer

11. (a) Because $f'(x) > 0$ when $0 < x < 2$, f is increasing on the interval $(0, 2)$.

2 pts: answer with reason $\left[\text{appeals to } f'(x) > 0\right]$

Reminder: In your reason, be sure to explicitly identify each function by name. Referring to $f'(x)$ as "it," "the function," or "the graph" may not receive credit on the exam.

(b) Because $f''(-0.8) = 0$ and $f''(1.3) = 0$, the graph of f has points of inflection at $x = -0.8$ and $x = 1.3$. The graph of f' is decreasing when $-2 < x < -0.8$ and $x > 1.3$, so the graph of f is concave downward on $(-2, -0.8)$ and $(1.3, \infty)$.

4 pts: $\begin{cases} \text{2 pts: intervals} \\ \text{2 pts: reason } \left[\text{appeals to where } f'(x) \text{ is decreasing}\right] \end{cases}$

Reminder: In your reason, be sure to explicitly identify each function by name.

(c) By the Mean Value Theorem,

$$\frac{f(-0.5) - f(0)}{-0.5 - 0} = f'(-0.5).$$

Because $f'(-0.5) < 0$, the slope of $f(x)$ is negative at $x = -0.5$.

3 pts: answer with reason

Note: In addition to reasoning with the Mean Value Theorem, an alternate explanation may involve justifying the sign of the numerator in the given difference quotient. Because $f'(x) < 0$ on $[-0.5, 0]$ from the given graph, f is decreasing on this interval. So, $f'(-0.5) > f(0)$, and the numerator of this difference quotient must be positive. With a positive numerator and negative denominator, the difference quotient itself must be negative.

12. (a) $f(x) = \dfrac{1 - 4x^2}{x} = \dfrac{1}{x} - 4x$

$f'(x) = \dfrac{1}{x^2} - 4$

f is decreasing when $f'(x) < 0$.

$0 = -\dfrac{1}{x^2} - 4$

$4 = -\dfrac{1}{x^2}$

$x^2 = -\dfrac{1}{4}$

$x = \pm\dfrac{1}{2}$

Use a table to test the critical numbers $x = \pm\dfrac{1}{2}$ and $x = 0$, where f' does not exist.

Interval	$-\infty < x < -\frac{1}{2}$	$-\frac{1}{2} < x < 0$
Test value	$x = -1$	$x = -\frac{1}{4}$
Sign of $f'(x)$	$f'(-1) = -\frac{1}{2} < 0$	$f'\left(-\frac{1}{4}\right) = -20 < 0$
Graph of f	Decreasing	Decreasing

Interval	$0 < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Test value	$x = \frac{1}{4}$	$x = 1$
Sign of $f'(x)$	$f'\left(\frac{1}{4}\right) = -20 < 0$	$f'(1) = -5 < 0$
Graph of f	Decreasing	Decreasing

So, f is decreasing on $(-\infty, 0)$ and $(0, \infty)$ because $f'(x)$ is negative on these intervals.

5 pts:
- 1 pt: computes $f'(x)$
- 1 pt: finds both critical numbers [sets $f'(x) = 0$ and solves]
- 1 pt: justification [examines sign of $f'(x)$ on either side of each critical number]
- 2 pts: answer with reason [appeals to where $f'(x) < 0$]

Notes: For the justification in this particular example, you could simply identify that $f'(x)$ is negative for all nonzero x-values. A sign chart may not be necessary.

If using a sign chart as part of the justification, the functions $f'(x)$ and f must be explicitly labeled in your chart. Unlabeled sign charts may not receive credit on the exam.

A sign chart alone is generally *not* sufficient for the explanation. To receive full credit on the exam, be sure to explain the information contained in the sign chart.

(b) f is concave downward when $f''(x) < 0$.

$f'(x) = -\dfrac{1}{x^2} - 4$

$f''(x) = \dfrac{2}{x^3}$

$f''(x) < 0$ when $x < 0$. So, $f(x)$ is concave downward on $(-\infty, 0)$.

3 pts:
- 2 pts: computes $f''(x)$ and examines where $f''(x) < 0$
- 1 pt: answer

(c) Because $f''(x) \neq 0$, the graph of f does not have any points of inflection.

1 pt: answer with reason

Reminder: In these explanations, be sure to explicitly identify each function by name. Referring to $f'(x)$ or $f''(x)$ as "it," "the function," or "the graph" may not receive credit on the exam.

13. (a) Use $f'(-3) = 0$ and $f'(0) = -1$ to find an equation of $f'(x)$ on $[-3, 0]$.

$$m = \frac{-1 - 0}{0 - (-3)} = -\frac{1}{3}$$

$$y - 0 = -\frac{1}{3}[x - (-3)]$$

$$y = -\frac{1}{3}x - 1$$

So, $f'(-1) = -\frac{1}{3}(-1) - 1 = -\frac{2}{3}$.

Because $m = -\frac{1}{3}$ at $f'(-1)$, $f''(-1) = -\frac{1}{3}$.

2 pts:
- 1 pt: finds $f'(-1)$ with justification (finds/uses an *equation* of the given line segment)
- 1 pt: finds $f''(-1)$ with justification (finds/uses the *slope* of the given line segment)

(b) On the interval $(-5, 0)$, $f'(-3) = 0$.

So, the graph has a critical number at $x = -3$.

Because the interval is open, the endpoints cannot be relative extrema. So, $x = -3$ is a relative maximum because $f'(x) > 0$ on $(-5, -3)$ and $f'(x) < 0$ on $(-3, 0)$.

1 pt: answer with reason [identifies that $f'(x)$ changes sign at $x = -3$]

(c) Possible points of inflection occur at $x = -4$, $x = 0$, and $x = 1$ because $f''(-4) = 0$ and $f''(0)$ and $f''(1)$ are undefined.

When $-5 < x < -4$, f' is increasing.
When $-4 < x < 0$, f' is decreasing.
When $0 < x < 1$, f' is increasing.
When $1 < x < 4$, f' is decreasing.

So, f is concave downward on the intervals $(-4, 0)$ and $(1, 4)$ because f' is decreasing.

2 pts: answers with reason [identifies where $f''(x) < 0$ by examining where $f'(x)$ is decreasing]

(d) The points of inflection occur at $x = -4$, $x = 0$, and $x = 1$ because $f''(-4) = 0$, $f''(0)$ and $f''(1)$ are undefined, and $f'(x)$ changes from either increasing to decreasing or decreasing to increasing at these x-values [see part (c)].

2 pts: answers with reason [identifies where $f''(x) = 0$ or where $f''(x)$ is undefined *and* that $f'(x)$ changes from increasing to decreasing or from decreasing to increasing at these x-values]

(e)
$$g(x) = f(x) + \sin^2 x$$
$$g'(x) = f'(x) + 2\sin x \cos x$$
$$g'\left(-\frac{\pi}{4}\right) = f'\left(-\frac{\pi}{4}\right) + 2\sin\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{4}\right)$$
$$= f'\left(-\frac{\pi}{4}\right) + 2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = f'\left(-\frac{\pi}{4}\right) - 1$$

From the graph, $f'\left(-\frac{\pi}{4}\right)$ is negative. So, $g'\left(-\frac{\pi}{4}\right)$ is negative, which means that g is decreasing at $x = -\frac{\pi}{4}$.

2 pts:
- 1 pt: computes $g'\left(-\frac{\pi}{4}\right)$
- 1 pt: answer with reason [identifies that $g'\left(-\frac{\pi}{4}\right)$ is negative]

Reminder: In these explanations, be sure to explicitly identify each function by name. For example, referring to $f'(x)$ or $f''(x)$ as "it," "the function," or "the graph" may not receive credit on the exam.